In the following let $X$ be a normed space.

**Reminder:** (cf. section II.3 in [Werner, Funktionalanalysis, 2007, 6. Auflage Springer])

An operator $T : X \to X$ is compact if for every bounded sequence $(x_n) \subset X$ the sequence $(Tx_n)_n$ has a convergent subsequence.

**Exercise 20** (Spectrum and eigenvalues of operators)

(a) Compute the eigenvalues and the spectrum of

$$A : L^2(0,1) \to L^2(0,1), \quad (Ax)(t) := (t + 1)x(t).$$

(b) Let $B : \{u \in C^1([0,1]) | u(0) = 0 \} \to C([0,1])$ with $Bu = u'$.

Show that the spectrum of $B$ is empty, i.e. show that $\sigma(B) = \emptyset$.

(c) Compute the eigenvalues of the shift operator

$$L : \ell^2 \to \ell^2, \quad L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

**Exercise 21** (Linear independence of eigenvectors of compact operators)

Let $T : X \to X$ be a compact operator.

Show that eigenvectors corresponding to pairwise distinct eigenvalues are linearly independent.

**Exercise 22** (Lemma 1.4, Compactness of the closed unit ball)

Show that the closed unit ball $B = \{x \in X | \|x\| \leq 1 \}$ is compact if and only if $X$ has finite dimension.

**Hint:** Use the Riesz Lemma 1.3.

**Exercise 23** (Lemma 1.5, cf. Satz II.3.2 in [Werner, Funktionalanalysis, 2007, 6. Auflage Springer])

Let $T : X \to X$ be a compact operator and let $S : X \to X$ be a bounded operator.

Show that the operators $TS$ and $ST$ are compact.