In the following let $\Omega \subseteq \mathbb{R}^d$ be a domain with boundary $\Gamma$.

**Aufgabe 1** (Fundamental Lemma of Calculus of Variations)
Let $w : \Omega \to \mathbb{R}$ be continuous and let
\[ \int_{\Omega} w(x) \cdot v(x) dx = 0 \quad \text{for all } v \in C^\infty_c(\Omega). \]
Show that $w(x) = 0$ for all $x \in \Omega$.

**Hint:** You can assume (without proof) that there exists a function $v_r \in C^\infty(\mathbb{R}^d)$ with the following properties:
- $v_r(x) > 0$ for $\|x\|_2 \leq r$ and $v_r(x) = 0$ for $\|x\|_2 < r$ (compact support in the ball with radius $r$).
- $\int_{\Omega} v_r(x) dx = 1$ (mass not depending on $r$).

**Aufgabe 2** (Properties of the bilinear form)
We define the space $V := \{ v : \Omega \to \mathbb{R}, v \in C(\Omega) \text{ and piecewise } C^1(\Omega) \text{ with } v(x) = 0 \ \forall x \in \Gamma \}$. Furthermore define the bilinear form $a : V \times V \to \mathbb{R}$ by
\[ a(u, v) := \sum_{i=1}^d \int_{\Omega} \partial_{x_i} u(x) \cdot \partial_{x_i} v(x) dx. \]

a) Show that the bilinear form $a : V \times V \to \mathbb{R}$ defines a scalar product.

b) Let $V_N \subset V$ be a $N$-dimensional subset of $V$ with basis $\{ \varphi_1, \ldots, \varphi_N \}$. Show that the matrix $A = (a(\varphi_i, \varphi_j))_{i,j} \in \mathbb{R}^{N \times N}$ is symmetric and positive definite.

**Aufgabe 3** (On soap bubbles and Laplace’s problem)
Every child knows how much fun it is to dip a closed wire frame into soap solution and then whistling some soap bubbles. But what is the shape of the soap film in the inside of the wire frame after pulling it out of the soap solution? We treat this problem mathematically:

Let $\Omega \subset \mathbb{R}^2$ be an open subset of $\mathbb{R}^2$ with boundary $\Gamma$ and let $g \in C(\Gamma)$, $g : \Gamma \to \mathbb{R}$ ($g$ describes the wire frame). We suppose that the surface of the soap film is given by the graph of $(x_1, x_2) \mapsto (x_1, x_2, u(x_1, x_2))$, where $u : \Omega \to \mathbb{R}$ is a smooth function.

The area of this surface is
\[ I(u) = \int_{\Omega} \sqrt{1 + (\partial_{x_1} u(x_1, x_2))^2 + (\partial_{x_2} u(x_1, x_2))^2} dx. \]

For the shape of the soap solution we furthermore require that $u(x) = g(x)$ on $\Gamma$ and that its area is minimal, i.e. $u$ satisfies
\[ u \in W := \{ \tilde{u} : \Omega \to \mathbb{R} \mid \tilde{u} \in C^2(\Omega) \cap C(\Omega), \quad I(\tilde{u}) \overset{!}{=} \min \quad \text{and} \quad \tilde{u} \mid_\Gamma = g \}. \]

Show that every $u \in W$ is also a solution of Laplace’s problem, i.e.
\[ -\Delta u = 0 \quad \text{in } \Omega \]
\[ u = g \quad \text{on } \Gamma. \]

Discussion in the problem class Monday 11:30, in room 3.061 in building 20.30.