2. Galerkin Method and Finite Element Space

Instead of the original problem (8), we approximately consider the variational formulation.

Find \( u \in V \) such that \( a(u,v) = l(v) \) for all \( v \in V \) \hspace{1cm} (2.1)

Remark: The space \( V \) will later be replaced by some suitable space.

Construction of a numerical method: Choose \( N \)-dimensional subspace \( V_N \subseteq V \) and seek an approximation \( u_N \approx u \) with the property

\[ a(u_N, v_N) = l(v_N) \quad \text{for all} \quad v_N \in V_N \] \hspace{1cm} (2.2)

Equivalent: \( u_N \in V \) such that \( J(u_N) \leq J(u) \) for all \( u \in V_N \)

(adopt previous proof (iii) \( \Rightarrow \) (iii) )

Since \( v_N \in V_N \subseteq V \), it follows that

\[ a(u_N - u, v_N) = a(u_N, v_N) - a(u, v_N) = l(v_N) - l(u_N) = 0 \]

Hence, the approximation \( u_N \) has the property that the error \( u_N - u \) is orthogonal with respect to \( a(\cdot, \cdot) \) to all \( v_N \in V_N \).

This property is called Galerkin condition.
Let \( \{q_1, \ldots, q_N\} \) be a basis of \( V_N \), \( \varphi : \Omega \to \mathbb{R} \).

Let
\[
U_N(x) = \sum_{i=1}^N \hat{u}_i q_i(x) \quad \text{and} \quad V_N(x) = \sum_{i=1}^N \hat{v}_i q_i(x)
\]
be representations of \( u_N \) and \( v_N \) in this basis. Substitute and use (bi-) linearity
\[
a(u_N, v_N) = \sum_{i=1}^N \sum_{j=1}^N \hat{u}_i \hat{v}_j a(q_i, q_j)
\]

(2.2)
\[
l(v_N) = \sum_{j=1}^N \hat{v}_j l(q_j) \quad \forall v_N \in V_N
\]

In Matrix-Vector notation this is equivalent to
\[
\hat{v}^T A \hat{u} = \hat{v}^T b \quad \forall \hat{v} \in \mathbb{R}^N
\]

(2.3)

with \( \hat{v} = \begin{pmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_N \end{pmatrix} \), \( \hat{u} = \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{pmatrix} \), \( A = (a(q_i, q_j)) \in \mathbb{R}^{N \times N} \), \( b = \begin{pmatrix} l(q_1) \\ \vdots \\ l(q_N) \end{pmatrix} \)

Since (2.3) must hold for all \( \hat{v} \in \mathbb{R}^N \), (2.3) is equivalent to the linear system
\[
A \hat{u} = b
\]

Thus (2.2) since \( a(\cdot, \cdot) \) is a scalar product, \( A \) is symmetric and positive definite. Thus, a unique solution \( \hat{u} \) exists, and we obtain the numerical approximation \( u_N \) from
\[
U_N = \sum_{i=1}^N \hat{u}_i q_i.
\]
3. Linear Elements

How to choose the ansatz space \( V_N \)? Consider \( d=2 \), \( (x_1, x_2) = (x, y) \)

(a) Triangulation

Approximate the boundary \( \Gamma \) by a polygon \( \tilde{\Gamma} \) with interior \( \tilde{\Omega} \).

![Diagram of a polygon approximating a boundary]

Saddle divide \( \tilde{\Omega} \) into \( N \) triangles. Vertices of a triangle must coincide with vertices of other triangles:

![Diagram of triangles sharing vertices]

Note: \( \Large \square, \quad \square \)

Notation: \( T_k \) \( k \)-th triangle (arbitrary enumeration)
\[ P_i = (x_i, y_i) \] vertices in \( \tilde{\Omega} \), i.e. \( P_i \in \tilde{\Gamma} \) \( i=1, \ldots, N \)

(b) Basis functions

Let \( \varphi_i \) be the hat function corresponding to \( P_i \), i.e.
\[ \varphi_i: T_k \text{ is linear on each } T_k \text{ and } \varphi_i(P_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases} \]

\[ \varphi_i: \tilde{\Omega} \rightarrow \mathbb{R} \] is continuous and piecewise linear

Choose \( V_N = \text{span} \{ \varphi_1, \ldots, \varphi_N \} \), seek approximation
\[ u_N(x, y) = \sum_{i=1}^{N} \hat{u}_i \varphi_i(x, y) \in V_N \]
Intepretation: \( u_N \in V_N \) is the piecewise linear interpolation of \( (p_j, \tilde{u}_j) \), because
\[
u_N(p_j) = \sum_{i=1}^n \tilde{u}_i \phi_i(p_j) = \tilde{u}_j.
\]

(c) **Stiffness matrix** \( A \) and vector \( b \)

The entries of the stiffness matrix \( A \) are
\[
a(q_i, q_j) = \sum_{k=1}^m \int_{T_k} \left( \partial_x q_i(x,y) \partial_x q_j(x,y) + \partial_y q_i(x,y) \partial_y q_j(x,y) \right) d\sigma(x,y)
\]
\[= a_{T_k}(q_i, q_j),
\]

\( A \) is sparse, because \( a(q_i, q_j) = 0 \) if \( \text{supp}(q_i) \cap \text{supp}(q_j) \) is a null set (empty, edge, vertex).

\( a_{T_k}(q_i, q_j) \) can be computed exactly because \( \partial_x q_i \big|_{T_k} \) and \( \partial_y q_i \big|_{T_k} \) is constant (and often zero) for all \( i, k \).

The entries of \( b \) are
\[
b(q_i) = \sum_{k=1}^m \int_{T_k} g(x,y) q_i(x,y) d\sigma(x,y)
\]
\[= b_{T_k}(q_i).
\]

The integral has to be approximated by quadrature.

Details: Lickert: "Einleitung in das Wissenschaftliche Rechnen"

(d) **Solving the linear system** \( A \mathbf{u} = \mathbf{b} \)

Direct methods (Cholesky) if \( A \) is small.

Iterative methods if \( A \) is large: Conjugate gradients, multigrid methods. ...
II. Finite element methods for elliptic boundary value problems

1. Cax-Milgram lemma

Question: Does the variational formulation of the problem (cf. Proposition 1.1) have a unique solution? Is there a unique \( u \in V \) such that
\[
a(u, v) = l(v) \quad \forall v \in V
\]
or equivalently
\[
J(u) \leq J(v) \quad \forall v \in V
\]

**Definition 1.1 (V-elliptic bilinear form)**

A symmetric bilinear form \( a: V \times V \rightarrow \mathbb{R} \) on a Hilbert space \( V \) with inner product \( \langle \cdot, \cdot \rangle \) is called \( V \)-elliptic if the following holds:

(i) \( a \) is bounded: There is a constant \( M < \infty \) such that
\[
|a(u, v)| \leq M \|u\|_V \|v\|_V \quad \text{for all } u, v \in V.
\]

(ii) \( a \) is coercive: There is a constant \( \alpha > 0 \) such that
\[
a(u, v) \geq \alpha \|u\|_V^2 \quad \text{for all } u \in V.
\]

Remarks: If \( a(\cdot, \cdot) \) is \( V \)-elliptic, then
\[
M \|u\|_V^2 \geq a(u, v) \geq \alpha \|u\|_V^2,
\]
i.e. the norms \( \| \cdot \|_V \) and \( \| \cdot \|_V^2 \) are equivalent, \( \|u\|_V \approx \|u\|_V^2 \).
\( \Rightarrow \) \( V \) is also a Hilbert space with respect to the inner product \( a(\cdot, \cdot) \).