4. Finite elements

Definition 4.1 (Finite element)

A finite element is a compact and connected subset $K \subset \mathbb{R}^d$ along with

1. nodes $z_1, \ldots, z_r \in K$, and
2. a finite dimensional vector space $\mathcal{P}$ of polynomials $p : K \to \mathbb{R}$ such that for arbitrary $c_i, \ldots, c_r \in \mathbb{R}$ the interpolation problem
   \[ p(z_i) = c_i, \quad i = 1, \ldots, r \]
   has a unique solution $p \in \mathcal{P}$.

Consequence: Every $p \in \mathcal{P}$ is uniquely determined by $c_1, \ldots, c_r$.
\[ \dim(\mathcal{P}) = r, \quad \mathcal{P} \cong \mathbb{R}^r \]

There is a basis of polynomials $\psi_i \in \mathcal{P}$ ("shape functions") such that
\[ \psi_i(z_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \]

"nodal basis".

Every $p \in \mathcal{P}$ can be represented as
\[ p(x) = \sum_{i=1}^{r} \hat{p}_i \psi_i(x), \quad \hat{p}_i = p(z_i) \]

Notation:

For $x \in \mathbb{R}^d$, $\alpha \in \mathbb{N}_0^d$ define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}$ and

- $\mathcal{P}_j(K) := \{ p : K \to \mathbb{R} : p(x) = \sum c_\alpha x^\alpha \text{ where } 1 \alpha_1 \leq j \} \quad \text{total degree } \leq j$
- $\mathcal{P}_{j\omega}(K) := \{ p : K \to \mathbb{R} : p(x) = \sum c_\alpha x^\alpha \text{ where } 1 \alpha_{\infty} \leq j \} \quad \text{degree } \leq j \text{ in each variable}
Examples for $d=2$

(a) Triangular elements

Every triangle $K$ can be mapped to the reference element $\hat{K} := \{(r,s) \in [0,1] \times [0,1] : r+s \leq 1\}$

$\varphi_K: \hat{K} \rightarrow K$, $\varphi_K(r,s) = P_1 + r(P_2 - P_1) + s(P_3 - P_1)$

- Linear element (cf. 1.3)
  
  $Z_1, Z_2, Z_3$ corners of the triangle $K$
  
  $z_3 = (9,1)$  $z_2 = (1,0)$  $z_4 = (0,1)$

  Basis on $\hat{K}$:
  
  $\hat{Q}_1(r,s) = 1-r-s$ \hspace{1cm} $\text{dim } \hat{B} = 3$

  $\hat{Q}_2(r,s) = r$

  $\hat{Q}_3(r,s) = s$

  $\hat{Q}_i(z_3) = \{i : i \neq j\}$

  Basis on $K \subseteq \mathbb{R}$:

  $Q_i(z) = Q_i(\varphi_K(r,s)) = \hat{Q}_i(r,s)$

- Quadratic elements
  
  $Z_1, \ldots, Z_6$ corners of $K$ plus midpoints of the edges

  $\hat{K}$ $z_2$ $z_6$ $z_5$

  $z_4$ $z_3$ $z_2$

  $\hat{Q}_4(r,s) = (1-r-s)(1-2r-2s)$

  $\hat{Q}_5(r,s) = 4s(1-r-s)$

  $\hat{B} = \hat{B}_2$, $\text{dim } \hat{B} = 6$