Remark: The error increases when $\gamma$ decreases. But if the triangulation is chosen in such a way that $\frac{\gamma}{3} \leq C$ for all triangles, then we obtain

$$\|v - \bar{v}\|_{H^2(\Omega)} \leq C \|v\|_{H^2(\Omega)} \quad \forall \bar{v}, \forall v \in H^2(\Omega)$$

Hence, triangles with small angles should be avoided.


Next step: Bound for the interpolation error in $H \cdot H_2(\Omega)$

Lemma 6.4

Under the assumptions of Lemma 6.3, we have

$$\|v - \bar{v}\|_{L_2(\Omega)} \leq C \|v\|_{H^2(\Omega)} \quad \forall \bar{v} \in H^2(\Omega)$$

Proof:

(a) Reference triangle $\hat{\Omega}$, smooth function

For $v \in C^\infty(\hat{\Omega})$ and $\mathbf{x} = (x,y) \in \hat{\Omega}$, we have

$$v(\mathbf{x}) - v(0) = \int_0^1 \frac{d}{dt} v(t \mathbf{x}) dt = \int_0^1 \sqrt{\mathbf{\nabla} v(t \mathbf{x})^T \mathbf{x}} dt$$

integrating by parts

$$= \left[ \mathbf{\nabla} v(t \mathbf{x})^T \mathbf{x} \right]_0^1 - \int_0^1 \mathbf{x} \mathbf{\nabla}^2 v(t \mathbf{x})^T \mathbf{x} dt$$

$$= \mathbf{\nabla} v(0)^T \mathbf{x}$$
The corresponding equation for \( \hat{\nabla} \) reads

\[
\hat{\nabla} \nu(\xi) - \hat{\nabla}(0) = (\nabla)^{-1} \hat{\nabla} \nu(\xi) \nabla \xi
\]

because \( \hat{\nabla} \nu \in \mathcal{H}_2(\hat{\Gamma}) \), and hence \( \nabla^2 \hat{\nabla} \nu = 0 \).

\[
\Rightarrow \| \nabla - \hat{\nabla} \|_{L^2(\Omega)} \leq \| \nabla \nu - \hat{\nabla} \nu \|_{L^2(\Omega)} + \left( \int_0^1 \left( \int_{\Gamma} \left( \nabla^2 \nu(\xi) \right)^2 d\xi \right) d\tau \right)^{1/2}
\]

(\(*\ast\))

For every \( z \in \mathbb{R}^2 \) with \( k_2 H \leq 1 \) and every symmetric \( M \in \mathbb{R}^{2 \times 2} \) we have that

\[
|z^TMz| \leq \|z\|_F \cdot \|Mz\|_F \leq \|M\|_F \cdot \|z\| = \|M\|_F
\]

Apply this with \( z \leftrightarrow \xi \) and \( M \leftrightarrow \nabla^2 \nu(\xi) \) to the second term:

\[
\int_0^1 \left( \int_{\Gamma} \left( \nabla^2 \nu(\xi) \right)^2 d\xi \right) d\tau \leq \| \nabla^2 \nu(\xi) \|_F^2
\]

\[
\phi = \xi \Rightarrow \int_0^1 \int \| \nabla^2 \nu(\xi) \|_F^2 d\xi d\tau \\
= \int_0^1 \int \| \nabla^2 \nu(\xi) \|_F^2 d\tau d\xi \\
d\xi = d^2 d\xi
\]

\[
= \int \int \| \nabla^2 \nu(\xi) \|_F^2 d\xi d\tau \leq C \| \nabla \nu \|^2_{H^2(\Gamma)}
\]

Substituting into the inequality (\(*\ast\)) gives

\[
\| \nabla - \hat{\nabla} \|_{L^2(\Omega)} \leq \| \nabla \nu - \hat{\nabla} \nu \|_{L^2(\Omega)} + C \| \nabla \nu \|_{H^2(\hat{\Gamma})} \leq C \| \nabla \nu \|_{H^2(\Omega)}
\]

\[
= \| \nabla - \hat{\nabla} \|_{H^2(\Omega)} \leq C \| \nabla \nu \|_{H^2(\Omega)}
\]

Lemma 6.2.
(6) Arbitrary triangle $K$, smooth $u$

Via the transform $\phi_K : \hat{K} \to K$, $\phi_K : (\xi) \mapsto \rho + \xi$, it can be shown that

$$||v - \Pi v||_{L^2(K)} \leq C h^2 ||v||_{H^2(K)}$$

(exercise, use (a) and proceed as in the proof of Lemma 6.3)

(c) It can be shown that $\Pi : H^2(K) \to C^0(K)$ is continuous.

Via a density argument, the assertion follows for all $v \in H^2(K)$.

All in all, we have proven the following result.

**Theorem 6.5**

Let $u$ be the solution of the elliptic boundary value problem described in Section 5. Let $u_h$ be the approximation obtained by the Goldman ansatz with piecewise linear elements on a triangulation $\mathcal{T}_h$.

For every $K \in \mathcal{T}_h$, we assume that $\text{diam}(K) \leq h$ and that the radius of the inner circle is not smaller than $\delta$.

If $\frac{h}{\delta} \leq \text{const}$ and if $u \in H^2(\hat{K})$, then the error bound

$$||u - u_h||_{H^1(K)} \leq C h ||u||_{H^2(K)}$$

holds with a constant $C$ independent of $u$, $h$, and $\mathcal{T}_h$.

Proof: Combine Lemma 6.3 and 6.4 with the arguments from Section 5.
Next goal: Error bound in $\| \cdot \|_{L^2(\Omega)}$ instead of $\| \cdot \|_{H^2(\Omega)}$.

Problem: The strategy from Section 5 does not work, because $\| \cdot \|_{L^2(\Omega)}$ is not equivalent to the energy norm $\| \cdot \|_E$.

**Definition 6.6**

The variational problem

Find $u \in V$ such that $a(u,v) = \int f v \, dx \quad \forall v \in V$

is called $H^2$-regular if for every $f \in L^2(\Omega)$ the solution $u$ is in $H^2(\Omega) \cap V$ and

$$\| u \|_{H^2(\Omega)} \leq C_2 \| f \|_{L^2(\Omega)}$$

with a constant $C_2$ independent of $f$.

It can be shown that the problem is $H^2$-regular if $\Omega$ is convex or has $C^2$-boundary and either pure Dirichlet or Neumann boundary conditions are posed.

**Theorem 6.7**

Consider the same situation as in Theorem 6.5 and assume in addition that the boundary value problem is $H^2$-regular. Then, the error bound

$$\| u - u_h \|_{L^2(\Omega)} \leq C_1 \| u \|_{H^2(\Omega)}$$

holds with a constant $C_1$ independent of $u$, $h$ and $f_h$. 