8. Error bounds for higher-order elements

Let \( K \subseteq \mathbb{R}^2 \) be a polyhedron, and let

\[
P^k_0(K) := \{ p : K \to \mathbb{R}^2 : p(x) = \sum_{l=1}^{k} c_l x^l \}
\]

be the set of polynomials of total degree \( \leq k \). Let \( \mathcal{W} \) be any \( k \geq 1 \).

**Theorem 8.1**

There is a constant \( C = C(K, k) \) such that for every \( v \in H^{k+1}(K) \)

\[
\inf_{p \in P^k_0} \| v - p \|_{H^{k+1}(K)} \leq C \| v \|_{H^{k+1}(K)}.
\]

**Proof:** Let \( R = \dim P^k_0 \) and choose points \( x_1, x_2, \ldots, x_R \) such that the mapping

\[
p \mapsto \left( p(x_1), \ldots, p(x_R) \right) \quad : \quad P^k_0 \to \mathbb{R}^R
\]

is bijective. Notation: \( \| v \|_{H^{k+1}(K)} = \| v \|_{H^{k+1}(K)} \).

Assume that the generalized Poincaré inequality

\[
(*) \quad \| v \|_{H^{k+1}(K)} \leq C \left( \| v \|_{H^{k+1}(K)} + \sum_{r=1}^{R} |v(x_r)| \right)
\]

holds for all \( v \in H^{k+1}(K) \).

(Sobolev embedding \( \Rightarrow H^{k+1}(K) \subseteq C(K) \) for \( k \geq 1 \)

\( \Rightarrow \) evaluations \( v(x_r) \) are well-defined.)
Let \( v \in H^{k+1}(K) \) and let \( p \in P_k \) be the unique interpolation polynomial, i.e. \( p(x_r) = v(x_r) \) \( \forall r = 1, \ldots, R = \dim(P_k) \). Applying (4) to \( v - p \) yields

\[
\|v - p\|_{H^{k+1}} \leq C \|v - p\|_{H^{k+1}} \to 0 = C \|v\|_{H^{k+1}}.
\]

It remains to show (4).

Assume that (4) is not true. Then, there is a sequence \( (w_m) \) in \( H^{k+1}(K) \) with

\[
\|w_m\|_{H^{k+1}} \geq m \left( \frac{\|w_m\|_{H^{k+1}}}{m} + \sum_{r=1}^R |w_m(x_r)| \right) \quad \forall m \in \mathbb{N}.
\]

The functions \( v_m := \frac{w_m}{\|w_m\|_{H^{k+1}}} \) have the properties

(i) \( v_m \in H^{k+1}(K) \), \( \|v_m\|_{H^{k+1}} = 1 \) (i.e. bounded)

(ii) \( \|v_m\|_{H^{k+1}} + \sum_{r=1}^R |v_m(x_r)| \leq \frac{1}{m} \to 0 \) for \( m \to \infty \)

Rellich (Theorem 7.5) \( \Rightarrow \) \( v_m \) has a \( \|H^k \)-convergent subsequence \( (\tilde{v}_m) \), and \( \|\tilde{v}_m\|_{H^{k+1}} \to 0 \) due to (ii).

Since \( \|v\|_{H^{k+1}} = \|v\|_{H^{k+1}}^2 + \|v\|_{H^{k+1}}^2 \), \( \tilde{v}_m \) is a Cauchy sequence in \( H^{k+1} \) \( \Rightarrow \) \( \tilde{v}_m \) converges in \( H^{k+1} \), i.e. \( \lim_{m \to \infty} \tilde{v}_m = v \in H^{k+1}(K) \)

By construction \( \|v\|_{H^{k+1}} = 0 \), i.e. \( \exists \varepsilon > 0 \) \( \forall \varepsilon \in \mathbb{N}, \text{with } \varepsilon > k+1 \).
Via a density argument, it can be shown that \( v \in \mathbb{P}_k \).

Next, we consider the point evaluation

\[
E_r : H^{k+1} \rightarrow \mathbb{R}, \quad \nu \mapsto \nu (x_r)
\]

\( E_r \) is continuous because \( H^{k+1} (\Omega) \rightarrow C (\Omega) \) is continuous.

\[
\Rightarrow \lim_{m \rightarrow \infty} E_r (\tilde{v}_m) = E_r (v) = \nu (x_r)
\]

Moreover, (\( ii \)) implies that

\[
\lim_{m \rightarrow \infty} E_r (\tilde{v}_m) = \lim_{m \rightarrow \infty} \tilde{v}_m (x_r) = 0.
\]

\[\Rightarrow \nu (x_r) = 0 \quad \forall r = 1, \ldots, R \quad \text{and} \quad v \in \mathbb{P}_k \implies v \equiv 0. \]

Remark: The proof does not give any information about the constant \( C \).

Now we can ready to investigate the interpolation error on the reference element \( \hat{\Omega} \).

**Theorem 8.2 (Bramble-Hilbert Lemma)**

Let \( \hat{\Omega} \) be the reference triangle. For \( m \in \mathbb{N} \) let \( \Pi : H^{k+1} (\hat{\Omega}) \rightarrow H^m (\hat{\Omega}) \) be linear and continuous, and let \( \Pi \nu = \nu \) for all \( \nu \in \mathbb{P}_k \).

(This is in particular true if \( \Pi \) is the finite element interpolation.)

Then, there is a constant \( C = C (k) \) such that

\[
\| \nu - \Pi \nu \|_{H^m (\hat{\Omega})} \leq C \| \nu \|_{H^{k+1} (\hat{\Omega})} \quad \forall \nu \in H^{k+1} (\hat{\Omega}).
\]
Proof: \( \Pi : H^k(\Omega) \rightarrow H^m(\Omega) \) is linear and continuous.

\[ \Rightarrow I - \Pi : H^{k+1}(\Omega) \rightarrow H^m(\Omega) \] is bounded.

Hence, for arbitrary \( p \in P_k \) and \( v \in H^{k+1}(\Omega) \) we have

\[ \| v - \Pi v \|_{H^m(\Omega)} = \| v - p - \Pi (v - p) \|_{H^m(\Omega)} \leq \| I - \Pi \|_{H^{k+1} \rightarrow H^m} \cdot \| v - p \|_{H^{k+1}} \]

\[ =: c_n \]

Since this is true for all \( p \in P_k \), it follows that

\[ \| v - \Pi v \|_{H^m(\Omega)} \leq c_n \inf_{p \in P_k} \| v - p \|_{H^{k+1}(\Omega)} \leq c_n \| v \|_{H^{k+1}(\Omega)} \]

Theorem 8.2

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Theorem 8.3 (Interpolation error)

Let \( K \) be an arbitrary triangle of the triangulation of \( \Omega \) with \( \text{diam}(K) = h \), radius \( g \) of the inner circle and nodal basis \( \varphi_r \), \( r = 1, \ldots, R \).

Let \( \hat{\Pi} : \hat{K} \rightarrow K \) be the affine linear bijective mapping from Lemma 6.3.

Let \( \Pi : H^{k+1}(K) \rightarrow P_k \) be the interpolation operator, i.e.

\[ \Pi v(x) = \sum_{r=1}^{R} v(Z_r) \varphi_r(x) \], \( Z_r \) nodes

Then

\[ \| v - \Pi v \|_{H^m(K)} \leq C \frac{h}{g^m} \| v \|_{H^{k+1}(K)} \]

for all \( v \in H^{k+1}(K) \) and \( m = 0, \ldots, k+1 \).
Proof: Transform to the reference element and use the Bramble–Hilbert lemma.

All in all, we have shown the following main result:

**Theorem 8.4** Error bound for finite elements of order $k$

Let $u$ be the solution of the elliptic boundary value problem described in section 5. Let $u_h$ be the approximation obtained by the Galerkin ansatz with finite elements with polynomial space $P_k$ on a triangulation $\mathcal{T}_h$.

For every $K \in \mathcal{T}_h$, we assume that diam$(K) \leq h$ and that the radius of the inner circle is not smaller than $g_0$.

(a) If $\frac{h}{g} \leq \text{const}$ and if $u \in H^{k+1}(\Omega)$, then the error bound

$$\|u - u_h\|_{H^k(\Omega)} \leq Ch^k \|u\|_{H^{k+1}(\Omega)}$$

holds with a constant independent of $u$, $h$, and $\mathcal{T}_h$.

(Cf. Theorem 6.5)

(b) If the problem is $H^2$-regular, then

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{k+1} \|u\|_{H^{k+1}(\Omega)}$$

Proof: Theorem 8.3 + $\frac{h}{g} \leq \text{const}$ yield

with $\text{max}_{\Omega} |\nabla u|$ the $L^2$-error

$$\|\nabla - \nabla_h\|_{L^2(\Omega)} \leq Ch^k \|\nabla u\|_{H^{k+1}(\Omega)}$$

$$\|\nabla - \nabla_h\|_{L^2(\Omega)} \leq Ch^{k+1} \|\nabla u\|_{H^{k+1}(\Omega)}$$

Together with section 5, this yields (a). (b) follows with Ritschel's trick.