6. Convergence of the W-cycle

Consider the multigrid method as in $\text{III.3}$ (W-cycle), but with
Richardson iteration instead of Gauss-Seidel (damped Jacobi).

The method approximates the solution $A^{(c)} \hat{u}^{(c)} = f^{(c)}$.
The following algorithm corresponds to one W-cycle:

\[
\hat{u}^{(c)}_{\text{new}} = \min \left( \hat{u}^{(c)}, \hat{u}^{(c-1)} \right)
\]

(i) Smoothing:
\[
\hat{v}^{(c)} = \hat{u}^{(c)}
\]
\[
\text{for } j = 0, \ldots, \nu - 1 \text{ do}
\]
\[
\hat{v}^{(c)}_{j+1} = \left( I - \frac{\nu}{a} A^{(c)} \right) \hat{v}^{(c)}_{j} + \frac{\nu}{a} \hat{u}^{(c)}_{j}
\]
\[
\text{end for}
\]
\[
\hat{v}^{(c)} := \hat{v}^{(c)}_{\nu}
\]

(ii) Defect:
\[
\hat{d}^{(c)} = f^{(c)} - A^{(c)} \hat{v}^{(c)}
\]

(iii) Restriction:
\[
\hat{d}^{(c-1)} = R^{(c-1)} \hat{d}^{(c)}
\]

(iv) Correction:
\[
\text{if } L > 1:
\]
\[
\hat{z}^{(c-1)} = \min \left( \hat{z}^{(c-1)}, \hat{d}^{(c-1)} \right)
\]
\[
\hat{z}^{(c-1)} = \min \left( \hat{z}^{(c-1)}, \hat{d}^{(c-1)} \right)
\]

\[
\text{else}
\]
\[
\text{solve } A^{(c)} \hat{z}^{(c)} = \hat{d}^{(c)}
\]
\[
\text{end}
\]
\[
\hat{u}^{(c)}_{\text{new}} = \hat{v}^{(c)} + \left( B^{(c-1)} \right)^T \hat{z}^{(c-1)}
\]

The new approximation $\hat{u}^{(c)}_{\text{new}}$ is better than $\hat{u}^{(c)}_{\text{old}}$.
We omit the additional smoothing after step (iv).
Let $g_l$ be the factor of error reduction of the multigrid method.

$$\| u_{\text{new}} - u \|_2 \leq g_l \| u_{\text{old}} - u \|_2$$

We have to show that $g_l \leq g$ for all $l = 1, \ldots, L$ with a constant $g < 1$ independent of $l$.

Idea: Use the convergence result of the two-grid method, cf. Section 5. In the two-grid method, however, the linear system on the coarser level is solved directly, cf. step (iv) in Section 5, whereas in step (iv) of mesh, the solution is approximated by another two calls of mesh unless $l = 1$. This difference is considered as an additional perturbation.

---

**Lemma 6.1**

Let $g_l$ be the reduction factor of the W-cycle with $l+1$ grids. If $l \geq 1$, then

$$g_l \leq g_1 + g_{l-1} (g_0 + g_1)$$

$$g_0 \leq \frac{c_0}{\gamma_0}, \quad \gamma_0 = \frac{c_0}{\gamma_0}$$

where $c_0$ and $C_0$ are the constants from Lemma 4.1.

**Proof:**

For $l > 1$ let $u_{\text{ex}}^{(l-1)}$ be the exact solution of $A^{(l-1)} u_{\text{ex}}^{(l-1)} = f^{(l-1)}$.

Define $g_i^{(l)} = u_{\text{ex}}^{(l-1)} + (s_i^{(l-1)})^T e_{\text{ex}}^{(l-1)}$  "exact update".

In step (iv) of mesh, $u_{\text{ex}}^{(l-1)}$ is approximated by $u_{\text{ex}}^{(l-1)}$, and the update

$$u_{\text{new}}^{(l)} = u_{\text{ex}}^{(l-1)} + (s_i^{(l-1)})^T e_{\text{ex}}^{(l-1)}$$

is used.
\[
\| u_{\text{new}}^{(r)} - u^{(e)} \|_2 \leq \| u_{\text{new}}^{(r)} - y^{(e)} \|_2 + \| y^{(e)} - u^{(e)} \|_2
\]  

(1)

Theorem 5.4 yields
\[
\| y^{(e)} - u^{(e)} \|_2 \leq s_n \left( \| u_{\text{old}}^{(e)} - u^{(e)} \|_2 + s_n \frac{c}{K} \right)
\]  

(2)

which we will prove that
\[
\| u_{\text{new}}^{(e)} - y^{(e)} \|_2 \leq s_n \left( \| u_{\text{old}}^{(e)} - u^{(e)} \|_2 + \frac{c}{K} \right)
\]  

(3)

Substituting (2) + (3) in the right-hand side of (1) yields the assertion.
\[
\| u_{\text{new}}^{(e)} - y^{(e)} \|_2 = \| u_{\text{new}}^{(e)} - (u_{\text{old}}^{(e)} + \varepsilon^{(e)}) \|_2 = \| (u_{\text{new}}^{(e)} - \varepsilon^{(e)}) - \varepsilon^{(e)} \|_2 
\]  

\[
\leq s_n \| \varepsilon^{(e)} + \varepsilon^{(e)} \|_2 \leq s_n \| 0 - \varepsilon^{(e)} \|_2 = s_n \| u_{\text{old}}^{(e)} - u^{(e)} \|_2
\]  

(4)

If we choose \( s_n \) in such a way that \( s_n \approx \max (A^{(e)}) \), then smoothing with Richardson yields
\[
\| \hat{v}^{(e)} - u^{(e)} \|_0 = \| (I - \frac{\tau}{\lambda^2} A^{(e)}) v^0 (\hat{u}_{\text{old}}^{(e)} - u^{(e)}) \|_0 \leq \| \hat{u}_{\text{old}}^{(e)} - u^{(e)} \|_0
\]  

(4.1)

and Lemma 4.1 gives
\[
\| u^{(e)} - u^{(e)} \|_2 \leq \gamma_0 \| u_{\text{old}}^{(e)} - u^{(e)} \|_2 .
\]  

(5)

This is now used to prove a bound for \( \| \varepsilon^{(e)} \|_2 \) in (4),
\[
\| \varepsilon^{(e)} \|_2 \leq \| y^{(e)} - u^{(e)} \|_2 + \| u^{(e)} - v^{(e)} \|_2 
\]  

\[
= \varepsilon^{(e)} + \varepsilon^{(e)}
\]

\[
\leq \left( s_n + \gamma_0 \right) \| u_{\text{old}}^{(e)} - u^{(e)} \|_2
\]  

(6)

Substituting (6) into (4) yields (2). This finishes the proof.
Theorem 6.2 (Convergence of the W-cycle)

Let $\gamma_0$ be the reduction factor of the W-cycle. If

$$\gamma_n \leq \frac{1}{2}$$

with

$$\gamma_n < \frac{\gamma_0}{\gamma_0 + \gamma_0} < \frac{1}{4}$$

($\gamma_0 = \frac{C_0}{c_0}$; cf. Lemma 6.1)

then the convergence rate of the W-cycle is bounded by

$$\gamma_l \leq \frac{1}{2}$$

for $l = 1, 2, 3, \ldots, L$

with

$$\gamma_l := \frac{1 - \sqrt{1 - 4 \delta (\delta + \gamma_0)}}{2 (\delta + \gamma_0)}$$

Proof: Since $\gamma_0 = \frac{C_0}{c_0} \geq 1$, it is clear that $\gamma \leq \frac{1}{2}$.

Prove $\gamma_l \leq \gamma$ by induction over $l$.

---

**l=1:** By assumption $\gamma_1 \leq \frac{1}{2}$. Show that $\delta \leq \gamma$:

$$\frac{\delta}{\gamma} = \frac{1 - \sqrt{1 - 4 \delta (\delta + \gamma_0)}}{2 \delta (\delta + \gamma_0)} = \frac{1 - \sqrt{1 - 4 \gamma_0}}{2 \gamma_0}$$

with $x_0 := \delta (\delta + \gamma_0) < \frac{1}{4}$

by assumption

For all $x \leq \frac{1}{4}$ we have

$$(1 - 2x)^2 = 1 - 4x + 4x^2 \geq 1 - 4x \geq 0$$

$\Rightarrow 1 - 2x \geq \sqrt{1 - 4x} \Rightarrow 1 - \sqrt{1 - 4x} \geq 2x \Rightarrow \frac{\delta}{\gamma} \leq 1$

$\Rightarrow \gamma \leq \delta \leq \gamma_1$

**l=2**

Lemma 6.1 and the assumption $\gamma_1 \leq \frac{1}{2}$ yield

$$\gamma_1 \leq \gamma_1 + \gamma_0^2 (\gamma_0 + \gamma_1) \leq 5 + \gamma_2^2 (\gamma_0 + 5) = \gamma$$

because $\gamma$ is the root of the polynomial $\gamma^2 (\gamma_0 + 5) - \gamma + 5$.  

---

\[\boxed{\text{125}}\]
Unfortunately, convergence of the V-cycle cannot be shown with this technique. For the V-cycle, the factor $\frac{1}{2}$ in Lemma 6.1 is replaced by $\frac{2}{3}$, and $g$ is the root of the linear equation
\[ g(y_0 + \delta) - g + \delta = 0 \]
i.e. \[ g = \frac{\delta}{y_0 + \delta - 1} < 0 \] because $y_0 - 1 > 0$.

7. Convergence in the energy norm

Goal: Prove convergence of the multigrid method with V-cycle or W-cycle in the energy norm
\[ \| u \|_h^2 = \int a(v,v) = \sqrt{\langle 0, A^2 \rangle} = \| u \|_A. \]
As in 5. and 6. we consider the Richardson iteration
\[ \widehat{u}_{j+1} = (I - \frac{\omega_2}{\omega_1} A^{(m)}) \widehat{u}_j + \frac{\omega_1}{\omega_2} \widehat{u}_j^{(e)} \]
for the smoothing steps with relaxation parameter $\omega_2$. As before we assume that $\lambda_{\max}(A^{(m)}) \leq \omega_2 \leq \frac{C}{h_0^2}$. 

The smoothness of a function \( v^{(e)} \in V^{(e)} \) is measured by

\[
\mathcal{N}(\hat{v}) := \begin{cases} 
1 - \frac{\lambda}{\omega_0} & \text{if } \|v^{(e)}\| \neq 0 \\
\lambda & \text{if } \|v^{(e)}\| = 0
\end{cases}
\]

Show that \( \mathcal{N}(\hat{v}) \in [0, 1] \):

Let \( \{q_1, \ldots, q_N\} \) be the eigendecay of \( A^{(e)} \), i.e., \( A^{(e)} q_i = \lambda_i q_i \), \( \|q_i\| = 1 \)

\( \langle q_i, q_j \rangle = 0 \)

Every \( \hat{v} \) has a representation

\[
\hat{v} = \sum_{i=1}^{N} s_i q_i \quad \text{with } s_i \in \mathbb{R}
\]

\[ \Rightarrow (A^{(e)})^S \hat{v} = \sum_{i=1}^{N} \lambda_i s_i q_i \]

Hence, by definition of \( \|v\| \), it follows that

\[
\mathcal{N}(\hat{v}) = \frac{1}{\omega_0} \frac{\|v\|_2^2}{\|v\|_1^2} = \frac{1}{\omega_0} \frac{\sum s_i^2 \lambda_i}{\sum s_i^2} = \frac{\sum s_i^2 (\lambda_i/\omega_0)^2}{\sum s_i^2 (\lambda_i/\omega_0)} \leq 1
\]

\( \sum s_i^2 (\lambda_i/\omega_0) \in [0, 1] \)

Why can \( \mathcal{N}(\hat{v}) \) be considered as a measure for smoothness?