Show that \( W := \text{span} \{ w_k \} \) is dense in \( H = W \oplus W^\perp \), i.e. prove that \( W^\perp = \{ 0 \} \).

\[ f \in W^\perp \iff \langle f, w_k \rangle = 0 \quad \forall k \]
\[ \iff \langle f, v_k \rangle = 0 \quad \forall k \]
\[ \iff f \in V^\perp = \{ 0 \} \quad \text{because } V \text{ is dense in } (H, \langle \cdot , \cdot \rangle) \]
\[ \iff f = 0 \]

Define Rayleigh quotient of a bilinear form \( a(\cdot , \cdot) \) :

\[ s_a(v) := \frac{a(v,v)}{\langle v,v \rangle}, \quad 0 \neq v \in V \]

If \( T \) is the operator defined in Lemma 4.15, then

\[ s_{T,a}(v) := \frac{a(Tv,v)}{a(v,v)} = \frac{\langle v,v \rangle}{a(v,v)} = \frac{a(v,v)}{a(v,v)} = s_a(v) \]

Lemma 2.2: Relation between the Rayleigh quotient and the largest/smallest eigenvalue.

An improved result is the minimax principle of Courant and Fischer:

**Theorem 3.3 (Courant–Fischer)**

For every \( N \) let \( V_m := \{ V_m : V_m \text{ subspace of } V \text{ and dim } (V_m) = m \} \).

Under the assumptions of Theorem 3.2, we have for every \( \text{me} \in N \)

\[ \min_{V_m \in V_m} \max_{0 \neq v \in V_m} s_a(v) = \lambda_m. \tag{1} \]

**Proof:**

Situation as in Theorem 3.2 : \( \omega_k \in V \) eigenvectors, i.e. \( a(\omega_k, v) = \lambda_k \langle \omega_k, v \rangle \forall v \in V \).

\[ \{ \omega_k \} \text{ Hilbert basis of } (H, \langle \cdot , \cdot \rangle) \]

\[ V_k = \frac{\omega_k}{\sqrt{\lambda_k}}, \quad \{ V_k \} \text{ Hilbert basis of } (V, a(\cdot , \cdot)) \]

* Prove \( \lambda_k \leq \lambda_m \):

Consider \( w_m := \text{span} \{ \omega_1, \ldots, \omega_m \} \in V_m \). Every \( v \in V_m \) has the representation

\[ v = \sum_{k=1}^{m} \alpha_k \omega_k = \sum_{k=1}^{m} \alpha_k \sqrt{\lambda_k} V_k, \quad \alpha_k = \langle v, \omega_k \rangle \]
\[ a(v, v) = \sum_{k=1}^{m} \alpha_k \lambda_k^2 \sqrt{\lambda_k} = \sum_{k=1}^{m} \frac{\lambda_k}{\lambda_k} \delta_{k}\lambda_k \]

\[ = \frac{a(v, v)}{\langle v, v \rangle} = \frac{\sum_{k=1}^{m} \alpha_k^2 \lambda_k}{\sum_{k=1}^{m} \frac{\alpha_k^2}{\lambda_k}} \leq \max_{k=1}^{m} \lambda_k = \lambda_{\text{max}} \]

The maximum is attained for \( v = v_{\text{max}} \), i.e., \( g_a(v_{\text{max}}) = \lambda_{\text{max}} \).

Hence:

\[ \min_{V_m \in U_m} \max_{0 \neq v \in V_m} g_a(v) \leq \max_{V_m \in U_m} g_a(v) = \lambda_{\text{max}} \tag{2} \]

Prove "\( \geq \)":

Let \( V_m = \text{span} \{ q_1, \ldots, q_m \} \subset U_m \) be an arbitrary \( m \)-dimensional subspace of \( V \).

Show that \( V_m \cap W_{m-1}^\perp \neq \{ 0 \} \).

If \( v = \sum_{i=1}^{m} q_i \in V_m \), then

\[ \forall v \in W_{m-1}^\perp \iff \langle v, w_k \rangle = 0 \quad \forall k=1, \ldots, m-1 \]

\[ \iff \sum_{i=1}^{m} \langle q_i, w_k \rangle v_i = 0 \quad \forall k=1, \ldots, m \]

\[ \iff Mv = 0, \quad v := (v_1, \ldots, v_m)^T \in \mathbb{R}^m \]

\[ M_{k,i} := \langle q_i, w_k \rangle, \quad M := (M_{k,i})_{m \times m} \]

Since \( M \in \mathbb{R}^{(m-1) \times m} \) there is a non-trivial solution \( v \in \mathbb{R}^m \) and the corresponding \( 0 \neq v = \sum_{i=1}^{m} q_i v_i \in V_m \cap W_{m-1}^\perp \) has the representation

\[ \sum_{k=1}^{m} \alpha_k w_k = \sum_{k=1}^{m} \lambda_k \sqrt{\lambda_k} v_k, \quad \alpha_k = \langle v, w_k \rangle \]

Substituting into \( g_a(v) \) yields

\[ g_a(v) = \frac{\sum_{k=1}^{m} \alpha_k^2 \lambda_k}{\sum_{k=1}^{m} \lambda_k} \leq \lambda_{\text{max}} \quad \Rightarrow \quad \max_{\forall v \neq 0} g_a(v) \geq \lambda_{\text{max}} \quad \forall V_m \subset U_m \]

\[ \Rightarrow \min_{V_m \in U_m} \max_{0 \neq v \in V_m} g_a(v) = \lambda_{\text{max}} \tag{3} \]

\[ (2) + (3) \Rightarrow (1) \]
4. Galerkin approximation of eigenvalue problems

Eigenvalue problem in variational formulation: Find $u \in V$ such that
\[ a(u, v) = \lambda \langle u, v \rangle \quad \forall v \in V \]
(cf. Theorem 3.2).

Choose subspace $V_h \subset V$, $\dim(V_h) = N < \infty$ finite.

Galerkin approximation: Seek $\lambda_h > 0$ and $0 \neq u_h \in V_h$ such that
\[ a(u_h, v_h) = \lambda_h \langle u_h, v_h \rangle \quad \forall v_h \in V_h. \]

The solution $(\lambda_h, u_h)$ is called Rayleigh-Ritz approximation with respect to $V_h$. $\lambda_h$ is called Ritz value, $u_h$ is called Ritz vector.

By adopting the proof of Theorem 3.2 it can be shown that eigenvalues
\[ 0 < \lambda_{1h} \leq \lambda_{2h} \leq \ldots \leq \lambda_{Nh} \quad N = \dim(V_h) \]
of the Galerkin approximation exist, and that both $(V_h, \langle \cdot, \cdot \rangle)$ and $(V_h, a(\cdot, \cdot))$ have an orthonormal basis of eigenvectors.

Let $V_{m,h} := \{ V_m : V_m \text{ subspace of } V_h \text{ and } \dim(V_m) = m \}$ be the set of all $m$-dimensional subspaces of $V_h$.

Courant-Fischer (Theorem 3.3) yields
\[ \lambda_{mh} = \min_{V_m \in V_{m,h}} \max_{0 \neq v \in V_m} \frac{a(v, v)}{\|v\|^2} = \min_{V_m \in V_{m,h}} \max_{0 \neq v \in V_m} \frac{\langle v, v \rangle}{\|v\|^2} = \lambda_m \quad (1) \]

Computing a Ritz pair $(\lambda_h, u_h)$:

Choose basis $\{ \phi_1, \ldots, \phi_N \}$ of $V_h$. Consider the representation
\[ u_h = \sum_{j=1}^N \hat{u}_j \phi_j, \quad \hat{u}_j \in \mathbb{R} \text{ unknown}. \]

Then
\[ a(u_h, v_h) = \lambda_h \langle u_h, v_h \rangle \quad \forall v_h \in V_h \]
\[ \iff \sum_{j=1}^N \hat{u}_j \sum_{i=1}^N \hat{v}_i a(\phi_j, \phi_i) = \lambda_h \sum_{j=1}^N \hat{u}_j \langle \phi_j, \phi_h \rangle \quad \forall i = 1, \ldots, N \]
\[ = a(\phi_i, \phi_h) \]
\( \Rightarrow \quad A \hat{u} = \lambda_n M \hat{u} \) (generalized eigenvalue problem in \( \mathbb{R}^N \))

with stiffness matrix \( A = \{a(q_i, q_j)\}_{i,j=1}^N \)

mass matrix \( M = \{\langle q_i, q_j \rangle\}_{i,j=1}^N \)

(similar to approximation of boundary value problems)

Transformation to a standard eigenvalue problem:

The mass matrix is symmetric and positive definite. Cholesky: \( M = LL^T \)

\[
\Rightarrow \quad L^{-T} A L^{-T} (L^T \hat{u}) = L^{-T} A \hat{u} = \lambda_n L^{-T} \frac{L^T \hat{u}}{M} = \lambda_n L^T \hat{u}
\]

Hence, \( \hat{\xi} := L^T \hat{u} \) is the solution of the eigenvalue problem

\[
(L^{-T} A L^{-T}) \hat{\xi} = \lambda_n \hat{\xi}
\]