4. Partial differential equations on sparse grids

Model problem: Poisson equation

$$(1) \quad \begin{cases} -\Delta u = f & \text{on } \Omega = (0,1)^d \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Assumption: $f : \bar{\Omega} \to \mathbb{R}$ continuous with bounded mixed derivatives

$$D^{\alpha} f = \frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} f \in \underbrace{C(\Omega)}_{\text{continuous functions}} \quad \text{for } |\alpha|_\infty \leq 2$$

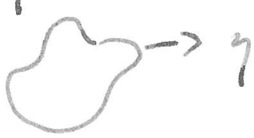
Let $u \in C^2(\Omega) \cap C_0(\Omega)$ be a classical solution of (1).

$$\Rightarrow \quad -\int_\Omega \Delta u \cdot v \, dx = \int_\Omega f \cdot v \, dx =: \langle f, v \rangle_{L^2(\Omega)}$$

for all $v \in C_0^\infty(\Omega)$

Apply Green's formula (cf. integration by parts)

$$-\int_\Omega \frac{\partial w}{\partial x_i} \cdot v \, dx = \int_\Omega w \frac{\partial v}{\partial x_i} \, dx - \int_{\partial\Omega} w \cdot v \, \eta_i \, ds$$



with $w = \frac{\partial u}{\partial x_i}$, $v \in C_0^\infty(\Omega)$. This yields

$$-\int_\Omega \frac{\partial^2 u}{\partial x_i^2} \cdot v \, dx = \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx - \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial x_i} \cdot v \, \eta_i \, ds}_{= 0 \text{ because } v|_{\partial\Omega} = 0}$$

$$\Rightarrow \quad -\int_\Omega \Delta u \cdot v \, dx = \sum_{i=1}^{d} \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} \, dx =: a(u,v)$$

$\Rightarrow$ Every classical solution satisfies

$$(2) \quad a(u,v) = \langle f, v \rangle_{L^2(\Omega)} \qquad \forall v \in C_0^\infty$$

The converse is also true: (2) $\Rightarrow$ (1) if $u \in C^2(\Omega) \cap C_0(\bar{\Omega})$
(use fundamental lemma)

This motivates the

Weak (or variational) formulation:

Find $u \in H := H_0^1(\Omega)$ such that

$$(3) \quad a(u,v) = \langle f, v \rangle_{L^2(\Omega)} \qquad \forall v \in H$$

The bilinear form $a : H \times H \longrightarrow \mathbb{R}$ is elliptic, i.e.

- $a(u,v) = a(v,u)$ (symmetric)
- $|a(u,v)| \leq C \cdot \|u\|_H \cdot \|v\|_H$ for some $C > 0$ (continuous)
- $a(v,v) \geq \alpha \|v\|_H^2$ for some $\alpha > 0$

Lax-Milgram lemma: A unique solution of (3) exists.

Numerical approximation:

Consider problem in a finite-dimensional space (here: sparse grid space): Find $\tilde{u}_n \in V_n^{(1)}$ such that

$$(4) \qquad a(\tilde{u}_n, v) = \langle f, v \rangle_{L^2(\Omega)} \qquad \forall v \in V_n^{(1)}$$

(Galerkin condition)

Equation (4) is equivalent to a linear system for the coefficients of $\tilde{u}_n$.

LEMMA 7 (Céa's lemma)

Let $\|\cdot\|_E$ be the energy norm, i.e. $\|v\|_E = \sqrt{a(v,v)}$ $\forall v \in H$. Let u and $\tilde{u}_n$ be the solution of (3) and (4), respectively. Then

$$\|u - \tilde{u}_n\|_E \leq \|u - v_n\|_E \qquad \forall v_n \in V_n^{(1)}$$

$\Rightarrow$ $\tilde{u}_n$ is the best-approximation in the energy norm.

Proof: Let $v_n \in V_n^{(1)} \subseteq H$

$$a(u, v_n) \overset{(3)}{=} \langle f, v_n \rangle_{L^2(\Omega)} \overset{(4)}{=} a(\tilde{u}_n, v_n)$$

$$\Rightarrow \quad 0 = a(u - \tilde{u}_n, v_n)$$

In particular (replace v_n by $\tilde{u}_n$) $\quad 0 = a(u - \tilde{u}_n, \tilde{u}_n)$

$$\begin{aligned} \Rightarrow \quad \|u - \tilde{u}_n\|_E^2 &= a(u - \tilde{u}_n, u - \tilde{u}_n) \\ &= a(u - \tilde{u}_n, u) \\ &= a(u - \tilde{u}_n, u - v_n) \\ &\overset{\text{Cauchy-}}{\underset{\text{Schwarz}}{\leq}} \|u - \tilde{u}_n\|_E \cdot \|u - v_n\|_E \end{aligned}$$

$$\Rightarrow \quad \|u - \tilde{u}_n\|_E \leq \|u - v_n\|_E$$

1.12.05

$\blacksquare$

COROLLARY

If u and $\tilde{u}_n$ are the solutions of (3) and (4), respectively, then

$$\|u - \tilde{u}_n\|_E \leq C_1(d) \cdot |u|_{2,\infty} \cdot 2^{-n}$$

$$\|u - \tilde{u}_n\|_E \leq C_2(d) \cdot |u|_{2,2} \cdot 2^{-n}$$

with $C_i(d)$ depending only on d.

Proof: Let $u_n^{(1)} \in V_n^{(1)}$ be the (unknown) interpolation of u on the sparse grid.

Lemma 7 $\Rightarrow$ $\|u - \tilde{u}_n\|_E \leq \|u - u_n^{(1)}\|_E$

The result follows from Theorem 3. $\blacksquare$

5. Extensions

(a) Optimality:

The sparse grid space $V_n^{(1)}$ can be found by solving a restricted optimization problem: Find a space $V^{(opt)}$ such that

$$\max_{\substack{u \in X_0^{q,2} \\ |u|=1}} \|u - u_{V^{(opt)}}\| = \min_{\substack{U \subset V \\ |U| = w}} \max_{\substack{u \in X_0^{q,2} \\ |u|=1}} \|u - u_U\| \qquad (*)$$

(see B & G, p. 19). For $\|\cdot\| = \|\cdot\|_2$ or $\|\cdot\| = \|\cdot\|_\infty$, we obtain $V^{(opt)} = V_n^{(1)}$.

(b) Energy-based sparse grids:

$V_n^{(1)}$ is not the optimal choice for the energy norm. The optimal sparse grid space $V_n^{(E)} \subseteq V_n^{(1)}$ for the energy norm can be found by solving $(*)$ with $\|\cdot\| = \|\cdot\|_E$.

(see B & G, p. 31)

(c) Generalization to other problems and boundary conditions

(d) More sophisticated basis functions:

Higher-order polynomials, wavelets, interpolets, ...

(e) Adaptive sparse grids