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Highly Oscillatory Problems — Exercise Sheet 01 April 20, 2018

In the lecture, the nonlinear Klein–Gordon equation

$$c^{-2}\partial_{tt}z(t,x) - \Delta z(t,x) + c^2 z(t,x) = f(z(t,x)), \quad z(0,x) = \varphi(x), \quad \partial_t z(0,x) = c^2\gamma(x),$$

serves as a model equation in order to construct and analyze efficient numerical schemes for the time integration of highly oscillatory wave-type partial differential equations (PDEs). In order to better understand the ideas in the construction of these schemes, we firstly focus on the simpler case of a second order (nonlinear) ordinary differential equation (ODE).

For $T > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we thus consider the model ODE problem

$$y''(t) = -Ay(t) + \lambda f(y(t)), \quad y(0) = y_0 \in \mathbb{R}^N, \quad y'(0) = y_0' \in \mathbb{R}^N \qquad \text{for all} \quad t \in [0,T], \tag{1}$$

where $A \in \mathbb{R}^{N\times N}$ is a regular, diagonalisable matrix with eigenvalues $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_N \in \mathbb{R}$ and where $f : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function.

Exercise 1: (Exact Solution in the Linear 1-dimensional Case)

Let $N = 1$, $A = \omega^2$, $\omega \in \mathbb{R}_{>0}$ and let $f \equiv 0$.

(a) Show that the solution of (1) is given by

$$y(t) = \cos(\omega t)y_0 + \frac{\sin(\omega t)}{\omega}y_0', \qquad \text{for all } t \in [0,T].$$

(b) Show that $y(t)$ can be rewritten in the form

$$y(t) = \frac{1}{2}\left(e^{i\omega t}(y_0 - i\frac{1}{\omega}y_0') + e^{-i\omega t}(y_0 + i\frac{1}{\omega}y_0')\right) = \frac{1}{2}\left(e^{i\omega t}u_0 + e^{-i\omega t}\overline{u_0}\right),$$

where $u_0 = y_0 - i\frac{1}{\omega}y_0'$.

(c) Show that $u(t) := e^{i\omega t}u_0$ with u_0 given in part b) solves the first order ODE

$$iu'(t) = -\omega u(t), \quad u(0) = u_0. \tag{2}$$

(d) Show that $u(t)$ given in c) and its corresponding differential equation (2) can also be obtained via the transform

$$u(t) = y(t) - i\frac{1}{\omega}y'(t), \quad \text{for all} \quad t \in [0,T]. \tag{3}$$

Exercise 2: (Matrix Functions)

Let $A \in \mathbb{R}^{N\times N}$ be a diagonalisable matrix with $A = S^{-1}DS$, where $D = \text{diag}(d_1,\dots,d_N) \in \mathbb{R}^{N\times N}$ is a diagonal matrix containing the eigenvalues of A, and where $S \in \mathbb{R}^{N\times N}$ is regular. Show that

$$\exp(A) = S^{-1}\exp(D)S, \quad \text{where} \quad \exp(D) = \text{diag}(e^{d_1},\dots,e^{d_N}).$$

Hint: Taylor series expansion of the exponential function.

Remark: Note that the idea of writing

$$g(A) = S^{-1}g(D)S, \quad \textit{where} \quad g(D) = \text{diag}(g(d_1),\dots,g(d_N)).$$

works for all functions g, for which its corresponding Taylor series expansion of g converges in d_j, $j = 1,\dots,N$. Typical examples for g are any polynomial, cosine, sine, exponential, square root and many more functions.

Exercise 3: (Exact Solution in the Nonlinear N-dimensional Case)

Let $N \in \mathbb{N}$ and let us consider $f : \mathbb{R}^N \to \mathbb{R}^N$ which is defined pointwise, i.e.

$$f(y) := (f(y_1), \dots, f(y_N))^\top \qquad \text{for all } y \in \mathbb{R}^N.$$

(a) Apply the transform (3) for the variable $u(t)$ to our model problem (1), i.e.

$$y''(t) = -Ay(t) + f(y(t)), \quad y(0) = y_0 \in \mathbb{R}^N, \quad y'(0) = y_0' \in \mathbb{R}^N \qquad \text{for all} \quad t \in [0, T],$$

and derive a corresponding first order equation for u of type (2).

(b) Use Duhamel's formula in order to write down a solution $u(t)$ at time $t \in [0, T]$ to the latter first order equation derived in a).

Hint: Consider $A = \Omega^2$ for a suitable matrix Ω and exploit the basic results of exercise 2.

Programming Exercise 1: (Visualization of the High Oscillations in the Linear Case)

Consider the setting of exercise 1, i.e. let $N = 1$, $A = \omega^2$, $\omega \in \mathbb{R}_{>0}$ and let $f \equiv 0$.

Write a MATLAB or Python script file which plots the exact solution $y(t)$ of (1) on the interval $[0, T = 10]$ for $\omega = 1, 2, 5, 8, 10, 20$ and for initial data $y_0 = 1$, $y_0' = 0.5\omega$.

Arrange the plots in a "plot matrix" using the `subplot` command.

Hint: You should use at least 200 time points in order to resolve the oscillations.

Discussion in the problem class friday 11:30 am, in room 2.058 in the Kollegiengebäude Mathematik 20.30.