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**Highly Oscillatory Problems — Exercise Sheet 04** June 14, 2018

Consider the nonlinear Klein–Gordon equation with a smooth nonlinearity $f(z)$ satisfying $f(\overline{z}) = \overline{f(z)}$

$$\partial_{tt}z(t,x) = -\underbrace{c^2(-\Delta + c^2)}_{=(c\langle\nabla\rangle_c)^2} z(t,x) + c^2\lambda f(z(t,x)), \quad z(0,x) = \varphi(x), \quad \partial_t z(0,x) = c^2\gamma(x), \qquad c \gg 1 \tag{KG}$$

with solution $z(t,x) \in \mathbb{C}$ for $t \in [0,T]$ and $x \in [-\pi,\pi] =: \mathbb{T}$, equipped with periodic boundary conditions. Furthermore, recall that by the ansatz of writing (we leave out the explicit dependence on $x$ in the following)

$$\begin{cases} u(t) = z(t) - ic^{-1}\langle\nabla\rangle_c^{-1}\partial_t z(t), \\ v(t) = \overline{z}(t) - ic^{-1}\langle\nabla\rangle_c^{-1}\overline{\partial_t z}(t), \end{cases} \qquad z(t) = \tfrac{1}{2}\left(u(t) + \overline{v}(t)\right)$$

we derived the coupled first order in time equation for $u(t)$ and $v(t)$

$$\begin{cases} i\partial_t u(t) = -c\langle\nabla\rangle_c u(t) + \lambda c\langle\nabla\rangle_c^{-1} f\left(\tfrac{1}{2}\left(u(t) + \overline{v}(t)\right)\right), & u(0) = \varphi - ic\langle\nabla\rangle_c^{-1}\gamma, \\ i\partial_t v(t) = -c\langle\nabla\rangle_c v(t) + \lambda c\langle\nabla\rangle_c^{-1} f\left(\tfrac{1}{2}\left(\overline{u}(t) + v(t)\right)\right), & v(0) = \overline{\varphi} - ic\langle\nabla\rangle_c^{-1}\overline{\gamma}. \end{cases} \tag{1}$$

**Asymptotic ansatz for sufficiently smooth solutions:**

We have seen that the error of an exponential integrator applied to the highly oscillatory system (1) heavily depends on the large parameter $c^2 \gg 1$. However, classical analytical results (see for instance [Mas2002]) imply the asymptotic approximation result

$$z(t) = z_0(t) + \mathcal{O}\left(c^{-2}\right), \qquad z_0(t) = \tfrac{1}{2}\left(e^{ic^2 t}u_0(t) + e^{-ic^2 t}\overline{v_0}(t)\right), \qquad \text{i.e.} \qquad z(t) \to z_0(t),\ c \to \infty, \tag{2}$$

where according to [Mas2002] the ansatz functions $u_0, v_0$ satisfy the system of nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u_0(t) = \tfrac{1}{2}\Delta u_0(t) + \lambda \tfrac{1}{2\pi}\int_0^{2\pi} f\left(\tfrac{1}{2}\left(u_0(t) + e^{-2i\theta}\overline{v_0}(t)\right)\right) d\theta, & u_0(0) = \varphi - i\gamma, \\ i\partial_t v_0(t) = \tfrac{1}{2}\Delta v_0(t) + \lambda \tfrac{1}{2\pi}\int_0^{2\pi} f\left(\tfrac{1}{2}\left(v_0(t) + e^{-2i\theta}\overline{u_0}(t)\right)\right) d\theta, & v_0(0) = \overline{\varphi} - i\overline{\gamma}. \end{cases}$$

**Exercise 8:** (Cubic Nonlinearity in Terms of $u$ and $v$)

Consider a cubic nonlinearity $f(z) := |z|^2 z$.

(a) Verify that for $u, v \in \mathbb{C}$

$$f\left(\tfrac{1}{2}(u + \overline{v})\right) = \tfrac{1}{8}\left[\left(|u|^2 + 2|v|^2\right)u + \left(|v|^2 + 2|u|^2\right)\overline{v} + u^2 v + \overline{v}^2\overline{u}\right].$$

(b) Exploit the result from part a) in order to show that

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(\tfrac{1}{2}\left(u_0(t) + e^{-2i\theta}\overline{v_0}(t)\right)\right) d\theta = \tfrac{1}{8}\left(|u_0(t)|^2 + 2|v_0(t)|^2\right)u_0(t).$$

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$\langle\nabla\rangle_c := \sqrt{-\Delta + c^2}$

[Mas2002] - N. Masmoudi and K. Nakanishi, *From nonlinear Klein–Gordon equation to a system of coupled nonlinear Schroedinger equations*, Math. Ann. 2002

In the following, consider the cubic nonlinearity $f(z) = |z|^2 z$. Note, that the highly oscillatory part of the limit solution

$$z_0(t) = \frac{1}{2}\left(e^{ic^2 t} u_0(t) + e^{-ic^2 t}\overline{v_0}(t)\right) \tag{3}$$

is contained only in the phases $e^{\pm ic^2 t}$. Moreover, the corresponding nonlinear Schrödinger (NLS) limit system for the ansatz functions $u_0,\ v_0$

$$\begin{cases} i\partial_t u_0(t) = \frac{1}{2}\Delta u_0(t) + \lambda\frac{1}{8}\left(|u_0(t)|^2 + 2|v_0(t)|^2\right) u_0(t), & u_0(0) = \varphi - i\gamma, \\ i\partial_t v_0(t) = \frac{1}{2}\Delta v_0(t) + \lambda\frac{1}{8}\left(|v_0(t)|^2 + 2|u_0(t)|^2\right) v_0(t), & v_0(0) = \overline{\varphi} - i\overline{\gamma} \end{cases} \tag{NLS}$$

is independent of the large parameter $c$. We can thus numerically approximate the limit ansatz $z_0$ from (3) by applying standard time integration schemes to the Schrödinger system (NLS). In the following, consider the discretization $t_n = n\tau,\ n = 0, 1, 2, \ldots, \lfloor T/\tau \rfloor$ of the time interval $[0, T]$ with time step size $\tau < 1$.

**Exercise 9:** (Strang Splitting Limit Integrator)

Consider the splitting of (NLS) into the kinetic and potential subproblems (T) and (P), respectively,

$$\begin{cases} i\partial_t u_0 = \frac{1}{2}\Delta u_0, & u_0(0) = u_0^0, \\ i\partial_t v_0 = \frac{1}{2}\Delta v_0, & v_0(0) = v_0^0, \end{cases} \tag{T}$$

$$\begin{cases} i\partial_t u_0 = \lambda\frac{1}{8}\left(|u_0|^2 + 2|v_0|^2\right) u_0, & u_0(0) = u_0^0, \\ i\partial_t v_0 = \lambda\frac{1}{8}\left(|v_0|^2 + 2|u_0|^2\right) v_0, & v_0(0) = v_0^0. \end{cases} \tag{P}$$

(a) Discuss the benefit of solving the subproblems (T) and (P) seperately instead of solving the full nonlinear system (NLS).

(b) Write down the exact solution of the kinetic subproblem (T) explicitly, i.e.

$$\begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} =: \varphi_{\mathrm{T}}^t \begin{pmatrix} u_0^0 \\ v_0^0 \end{pmatrix}$$

exploiting the Fourier series representation of $u_0(t)$ and $v_0(t)$.

(c) Consider subproblem (P).

$\alpha$) Show that within subproblem (P) there holds

$$|u_0(t)|^2 = |u_0(0)|^2 \quad \text{and} \quad |v_0(t)|^2 = |v_0(0)|^2 \qquad \text{for all } t \in [0, T].$$

$\beta$) Write down the exact solution of the potential subproblem (P) explicitly, i.e.

$$\begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} =: \varphi_{\mathrm{P}}^t \begin{pmatrix} u_0^0 \\ v_0^0 \end{pmatrix}.$$

(d) Explicitly write down the Strang splitting scheme with step size $\tau$

$$\begin{pmatrix} u_0^{n+1} \\ v_0^{n+1} \end{pmatrix} = \Phi_{\mathrm{Strang}}^{\tau} \begin{pmatrix} u_0^n \\ v_0^n \end{pmatrix} := \left(\varphi_{\mathrm{T}}^{\tau/2} \circ \varphi_{\mathrm{P}}^{\tau} \circ \varphi_{\mathrm{T}}^{\tau/2}\right) \begin{pmatrix} u_0^n \\ v_0^n \end{pmatrix}$$

for numerically computing an approximation $(u_0^{n+1}, v_0^{n+1})$ to the exact solution $(u_0(t_{n+1}), v_0(t_{n+1}))$ at time $t_{n+1}$ of (NLS).

(e) Formulate a theorem on the convergence towards the exact solution $z(t)$ of the Klein–Gordon equation (KG) of the numerical approximation

$$z_0^n := \frac{1}{2}\left(e^{ic^2 t_n} u_0^n + e^{-ic^2 t_n}\overline{v_0^n}\right)$$

obtained via the Strang splitting scheme $\Phi_{\mathrm{Strang}}^{\tau}$.
Briefly structure and sketch the proof of this theorem.

***Hint:*** Exploit the analytical result (2) on the asymptotic behaviour of $z(t)$ towards $z_0(t)$.

**Programming Exercise 5:** (Implementation of the Limit Integrator)

In this programming exercise we numerically underline the asymptotic approximation property (cf. (2))

$$z(t) = z_0(t) + \mathcal{O}\left(c^{-2}\right) = \tfrac{1}{2}\left(e^{ic^2t}u_0(t) + e^{-ic^2t}\overline{v_0}(t)\right) + \mathcal{O}\left(c^{-2}\right), \qquad c \to \infty,$$

where $z$ solves (KG) with $f(z) = |z|^2 z$ and where the functions $(u_0, v_0)$ satisfy (NLS).

More precisely, we consider the Klein–Gordon equation (KG) with $\lambda = 0.5$ and the cubic nonlinearity $f(z) = |z|^2 z$ on the torus $\mathbb{T} = [-\pi, \pi]$ (i.e. periodic boundary conditions) and for $t \in [0, T]$. Furthermore we fix the (complex valued) initial data

$$\varphi(x) = \frac{\cos(x) + i\sin(2x)}{2 - \sin(x)}, \qquad \gamma(x) = \frac{\sin(2x) + i\cos(x)}{2 - \cos(x)}.$$

For practical implementation issues, we consider the discretization of space $x_j = jh,\ j = -\frac{N_x}{2} + 1, \ldots \frac{N_x}{2}$ with $N_x = 64$, i.e. $h = \frac{2\pi}{N_x}$, and of time $t_n = n\tau,\ n = 0, 1, 2, \ldots \lfloor T/\tau \rfloor =: N_T$ for time step sizes

$$\tau = \tau_m = \tfrac{T}{N_T^m}, \quad \text{where} \quad N_T^m, m = 1, 2, \ldots, m_{\max}$$

is the $m$-th divisor of the number $N_T^{m_{\max}} = 120$. All divisors of a number $N \in \mathbb{N}$ can be found in MATLAB (since version R2014b) via the function `divisors()`. Moreover, we consider the $c$ values $c_\ell = 1.45^\ell,\ \ell = 0, 1, \ldots, 10$.

(a) In the code from Programming Exercise 4, extend the exponential integration scheme $\Phi^\tau_{\exp}$ from Exercise Sheet 3 such that it applies also for (KG) with cubic nonlinearity, and in particular for complex initial data, i.e. according to system (1) implement the time stepping for the function $v$ additionally to the time stepping for the function $u$.

Because for the cubic Klein–Gordon equation on the torus $\mathbb{T}$ we do not have an exact solution available, we need to compute a reference solution in order to test our schemes. Thus, before performing the actual time stepping with step size $\tau_m,\ m = 1, 2, \ldots, m_{\max}$ for a specific choice of $c = c_\ell,\ \ell = 0, 1, \ldots, 10$, we carry out the time integration over the full interval $t \in [0, T]$ with a suitable **reference scheme** using a smaller time step size $\tau_{\text{ref}} := \frac{\min_m \tau_m}{M_{\text{ref}}} \ll \tau_m$ for $M_{\text{ref}} \in \mathbb{N}$. Since we choose $N_T^m$ to be a divisor of $N_T^{m_{\max}}$ and due to $\tau_m = \frac{T}{N_T^m}$, we have that

$$\frac{\tau_m}{\tau_{m_{\max}}} = \frac{\tau_m}{\min_m \tau_m} = \frac{N_T^{m_{\max}}}{N_T^m} \in \mathbb{N} \qquad \text{for all } m = 1, 1, 2, \ldots, m_{\max}. \tag{4}$$

(b) $\alpha$) Implement the adapted exponential integration scheme $\Phi^{\tau_{\text{ref}}}_{\exp}$ with $\tau_{\text{ref}} := \frac{\min_m \tau_m}{M_{\text{ref}}}$ for $M_{\text{ref}} = 200$ as a reference method into the code from part a).
Note that you **only need to save** the numerical approximations

$$z^n_{\text{ref}} = \frac{1}{2}\left(u^n_{\text{ref}} + \overline{v^n_{\text{ref}}}\right) \qquad \text{at times } t_n = n\tau_{m_{\max}},$$

due to the relation (4). The corresponding time grid then covers also the coarser grids corresponding to $\tau_m$ for all $1 \le m \le m_{\max}$.

$\beta$) Proceed as in Programming Exercise 4 in order to create order plots (in $c$ as well as in $\tau$) for the numerical approximations

$$z^n = \frac{1}{2}\left(u^n + \overline{v^n}\right) \qquad \text{at times } t_n = n\tau_m,$$

obtained with the scheme $\Phi^\tau_{\exp}$.

(c) Implement the Strang splitting limit integration scheme $\Phi^\tau_{\text{Strang}}$ from Exercise 9 applied to the limit system (NLS) into the code from part b) and create additional order plots for the approximations

$$z^n_0 = \frac{1}{2}\left(e^{ic^2t_n}u^n_0 + e^{-ic^2t_n}\overline{v^n_0}\right) \qquad \text{at times } t_n = n\tau_m$$

obtained with this scheme. As a reference solution, use the same data as in part b)$\alpha$).

*Discussion in the problem class friday 11:30 am, in room 2.058 in the Kollegiengebäude Mathematik 20.30.*