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1. Maxwell's equations

1.1. Short history of electromagnetism

1802 Gian Domenico Romagnosi:
Compass needle reacts to electric current
=> relationship electricity <=> magnetism
Did not receive much attention

1820 Hans Christian Ørsted:
Rediscovers this effect => sensation in the scientific world

1820 André-Marie Ampère:
Theoretical explanation: Electric currents generate magnetic fields.

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1831 Michael Faraday:
A changing magnetic field induces an electric current.
Constructs first electric dynamo.

1861/65 James Clerk Maxwell:
Unifies and formalizes these theories => Maxwell's equations
System of time-dependent partial differential equations (PDEs)
Foundation of classical electromagnetic theory
Light = electromagnetic wave

1886/87 Heinrich Rudolf Hertz:
Shows experimentally the existence of electromagnetic waves (in Karlsruhe)

1.2 Notation: Vector products and differential operators

(a) Vector products: For $u, v \in \mathbb{R}^3$ define

- scalar product: $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$
- cross product

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \in \mathbb{R}^3$$

Important property: $(u \times v) \cdot v = (u \times v) \cdot u = 0$

(b) Differential operators: For a smooth function $u = u(t, x)$,
$u: \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ and a smooth vector field
$\mathcal{F}: \mathbb{R}_+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ we define $\qquad (\mathbb{R}_+ := (0, \infty))$

- partial derivatives: $\partial_t u = \frac{\partial u}{\partial t}$, $\partial_i u = \frac{\partial u}{\partial x_i}$ $\quad (i = 1, 2, 3)$
- gradient:

$$\nabla u = \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \partial_3 u \end{pmatrix}$$

- divergence: $\operatorname{div} \mathcal{F} = \partial_1 \mathcal{F}_1 + \partial_2 \mathcal{F}_2 + \partial_3 \mathcal{F}_3 \quad (= \text{„}\nabla \cdot \mathcal{F}\text{"})$
- curl:

$$\operatorname{curl} \mathcal{F} = \begin{pmatrix} \partial_2 \mathcal{F}_3 - \partial_3 \mathcal{F}_2 \\ \partial_3 \mathcal{F}_1 - \partial_1 \mathcal{F}_3 \\ \partial_1 \mathcal{F}_2 - \partial_2 \mathcal{F}_1 \end{pmatrix} = \operatorname{rot} \mathcal{F} = \text{„}\nabla \times \mathcal{F}\text{"}$$

- Laplace operator:

$$\Delta u = \partial_1^2 u + \partial_2^2 u + \partial_3^2 u$$

$$\Delta \mathcal{F} = \begin{pmatrix} \Delta \mathcal{F}_1 \\ \Delta \mathcal{F}_2 \\ \Delta \mathcal{F}_3 \end{pmatrix}$$

1.3 Integral form and differential form of Maxwell's equations

Let $\Omega \subseteq \mathbb{R}^3$ be a domain (open, connected), $\mathbb{R}_+ := (0, \infty)$

Electromagnetic waves are described by four vector fields

$\mathcal{E}: \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^3$	electric field intensity	$\frac{V}{m}$
$\mathcal{H}: \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^3$	magnetic field intensity	$\frac{A}{m}$
$\mathcal{D}: \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^3$	electric displacement	$\frac{C}{m^2}$
$\mathcal{B}: \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^3$	magnetic field induction	T

subject to

$\mathcal{J}: \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^3$	electric current density	$\frac{A}{m^2}$
$\varrho: \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$	electric charge density	$\frac{C}{m^3}$

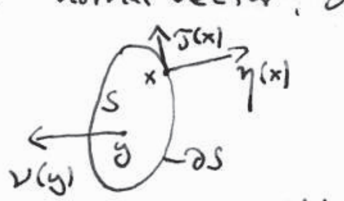
Setting:

$S \subseteq \Omega$ connected smooth surface with boundary ∂S

$\nu: S \longrightarrow \mathbb{R}^3$ unit normal vector, continuous, always directed to the same side of S („positive side")

$\tau: \partial S \to \mathbb{R}^3$ unit tangent vector of ∂S, directed counterclockwise when seen from the positive side of S.

$\eta: \partial S \to \mathbb{R}^3$ normal vector, directed to the outside of S



$V \subseteq \Omega$ open with boundary ∂V and outer unit normal vector $\nu: \partial V \longrightarrow \mathbb{R}^3$.

Maxwell's equations in integral form:

(1a) $$\int_{\partial S} \mathcal{H} \cdot \tau \, dl = \underbrace{\frac{d}{dt} \int_S \mathcal{D} \cdot \nu \, ds}_{\text{Maxwell's correction}} + \int_S \mathcal{J} \cdot \nu \, ds$$ Ampère's Circuital law

(1b) $$\int_{\partial S} \mathcal{E} \cdot \tau \, dl = -\frac{d}{dt} \int_S \mathcal{B} \cdot \nu \, ds$$ Faraday's law of induction

(2a) $$\int_{\partial V} \mathcal{D} \cdot \nu \, ds = \int_V \varrho \, dx$$ Gauss' electric law

(2b) $$\int_{\partial V} \mathcal{B} \cdot \nu \, ds = 0$$ Gauss' magnetic law

Interpretation:

(1a) A magnetic field can be generated by an electrical current or by a changing electric field.

(1b) A changing magnetic field induces an electric field.

(2a) A static electric field points away from positive charges to negative charges.

(2b) There are no magnetic monopoles („magnetic charges"): Magnetic field lines form loops – they never begin or end. Magnetic field lines which enter V must also exit V somewhere.

For sufficiently smooth vector fields $\mathcal{F} : \Omega \longrightarrow \mathbb{R}^3$, we have

- Stoke's theorem:

$$(3) \qquad \int_S \operatorname{curl} \mathcal{F} \cdot \nu \, ds = \int_{\partial S} \mathcal{F} \cdot \tau \, dl$$

- Gauss' divergence theorem

$$(4) \qquad \int_V \operatorname{div} \mathcal{F} \, dx = \int_{\partial V} \mathcal{F} \cdot \nu \, ds$$

Applying this to (1) and (2) yields ...

$$(1a') \qquad \frac{d}{dt} \int_S \mathcal{D} \cdot \nu \, ds \overset{(3)}{=} \int_S (\operatorname{curl} \mathcal{H} - \mathcal{J}) \cdot \nu \, ds$$

$$(1b') \qquad \frac{d}{dt} \int_S \mathcal{B} \cdot \nu \, ds \overset{(3)}{=} - \int_S \operatorname{curl} \mathcal{E} \cdot \nu \, ds$$

$$(2a') \qquad \int_V \operatorname{div} \mathcal{D} \, dx \overset{(4)}{=} \int_V \varrho \, dx$$

$$(2b') \qquad \int_V \operatorname{div} \mathcal{B} \, dx \overset{(4)}{=} 0$$

Since this is true for arbitrary S and V, we obtain ...

(11)

Maxwell's equations in differential form:

$$(5a) \qquad \partial_t \mathcal{D} = \operatorname{curl} \mathcal{H} - \mathcal{J}$$

$$(5b) \qquad \partial_t \mathcal{B} = -\operatorname{curl} \mathcal{E}$$

} PDEs

$$(6a) \qquad \operatorname{div} \mathcal{D} = \varrho$$

$$(6b) \qquad \operatorname{div} \mathcal{B} = 0$$

} divergence conditions

+ initial conditions + boundary conditions if $\Omega \neq \mathbb{R}^3$

The divergence conditions are in some sense redundant:

(12)

Lemma 1.1

Let $\mathcal{D}, \mathcal{B}, \mathcal{H}, \mathcal{E}$ be a smooth solution of (5a), (5b).
Suppose that ϱ and $\mathcal{J}$ satisfy the compatibility condition

$$(7) \qquad \partial_t \varrho + \operatorname{div} \mathcal{J} = 0,$$

and that (6a), (6b) are true for $t=0$, i.e.

$$\left.\begin{aligned} \operatorname{div} \mathcal{D}(0,x) &= \varrho(0,x) \\ \operatorname{div} \mathcal{B}(0,x) &= 0 \end{aligned}\right\} \quad \forall x \in \Omega.$$

Then, (6a) and (6b) are true for all $t \geq 0$.

16.4.