Proof: It can be checked that $\operatorname{div}(\operatorname{curl} \mathcal{F}) = 0$ for all smooth vector fields $\mathcal{F}: \Omega \longrightarrow \mathbb{R}^3$.

$$\Rightarrow \partial_t(\operatorname{div} \mathcal{D}) = \operatorname{div}(\partial_t \mathcal{D}) \overset{(5a)}{=} \underbrace{\operatorname{div}(\operatorname{curl} \mathcal{H})}_{=0} - \operatorname{div} \mathcal{J} \overset{(7)}{=} \partial_t \rho$$

$$\operatorname{div} \mathcal{D}(t,x) = \underbrace{\operatorname{div} \mathcal{D}(0,x)}_{\overset{\text{ass.}}{=} \rho(0,x)} + \int_0^t \underbrace{\partial_t \operatorname{div} \mathcal{D}(s,x)}_{= \partial_t \rho(s,x)} \, ds = \rho(t,x)$$

Moreover

$$\partial_t(\operatorname{div} \mathcal{B}) = \operatorname{div}(\partial_t \mathcal{B}) \overset{(5b)}{=} - \operatorname{div}(\operatorname{curl} \mathcal{B}) = 0$$

$$\Rightarrow \operatorname{div} \mathcal{B}(t,x) = \operatorname{div} \mathcal{B}(0,x) \overset{\text{ass.}}{=} 0 \qquad \blacksquare$$

2·3 PDEs for 4·3 functions $\Rightarrow$ need more equations

Constitutive relations

These equations model properties of the medium.

(a) In vacuum:

$$\mathcal{D}(t,x) = \varepsilon_0 \, \mathcal{E}(t,x) \qquad \varepsilon_0 \text{ permittivity of free space}$$

$$\mathcal{B}(t,x) = \mu_0 \, \mathcal{H}(t,x) \qquad \mu_0 \text{ permeability of free space}$$

($\varepsilon_0, \mu_0 > 0$ constants)

(b) Linear materials:

$$\mathcal{D}(t,x) = \varepsilon(x) \, \mathcal{E}(t,x) \qquad \varepsilon(x) \text{ electric permittivity}$$

$$\mathcal{B}(t,x) = \mu(x) \, \mathcal{H}(t,x) \qquad \mu(x) \text{ magnetic permeability}$$

$\varepsilon(x), \mu(x) \in \mathbb{R}^{3\times3}$ symmetric, positive definite

Can be discontinuous!

Isotropic materials: $\varepsilon(x), \mu(x) \in \mathbb{R}$ scalar valued

Homogeneous materials: ε, μ constant matrices / scalars

(c) General case

$$\mathcal{D} = \mathcal{D}(\mathcal{E}, \mathcal{H}), \qquad \mathcal{B} = \mathcal{B}(\mathcal{E}, \mathcal{H})$$

For example $\mathcal{D} = \varepsilon_0 \mathcal{E} + \mathcal{P}$ ($\mathcal{P}$ polarisation)

$\mathcal{B} = \mu_0 \mathcal{H} + \mu_0 \mathcal{M}$ ($\mathcal{M}$ magnetization field)

Example: $\mathcal{P} = \mathcal{P}(\mathcal{E}) = \varepsilon_0 (\varepsilon_r + \alpha |\mathcal{E}|^2) \mathcal{E}$ Kerr nonlinearity

constants

(d) Ohm's law in conducting media

$$\mathcal{J}(t,x) = \underbrace{\sigma(x)}_{\text{conductivity}} \mathcal{E}(t,x) + \underbrace{\mathcal{J}_e(t,x)}_{\text{external current density}}$$

In this lecture, we will focus on <u>linear materials</u>. Substituting $\mathcal{D} = \varepsilon \mathcal{E}$ and $\mathcal{B} = \mu \mathcal{H}$ into (5) and (6) yields the

<u>Linear Maxwell equations</u>

(8a) $\varepsilon \partial_t \mathcal{E} = \operatorname{curl} \mathcal{H} - \mathcal{J}$ PDEs

(8b) $\mu \partial_t \mathcal{H} = -\operatorname{curl} \mathcal{E}$

(9a) $\operatorname{div}(\varepsilon \mathcal{E}) = \rho$ divergence conditions

(9b) $\operatorname{div}(\mu \mathcal{H}) = 0$

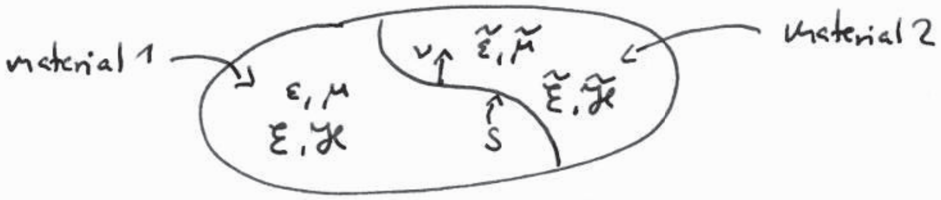
(10a) $\mathcal{E}(0,x) = \mathcal{E}^0(x)$ initial conditions

(10b) $\mathcal{H}(0,x) = \mathcal{H}^0(x)$

for some $\mathcal{E}^0, \mathcal{H}^0$ with $\operatorname{div}(\varepsilon \mathcal{E}^0) = \rho(0,\cdot)$, $\operatorname{div}(\mu \mathcal{H}^0) = 0$.

Interface conditions and boundary conditions

The functions $\varepsilon(x), \mu(x)$ can be discontinuous at boundaries between two different materials :



Question: Condition at the interface S?

It follows from Stoke's theorem (3) and the divergence theorem (4) that at the interface S we must have

$$(\mu \mathcal{H}) \cdot \nu = (\tilde{\mu} \tilde{\mathcal{H}}) \cdot \nu$$
$$\mathcal{E} \times \nu = \tilde{\mathcal{E}} \times \nu$$

Details: J.D. Jackson, „Classical electrodynamics"

Ohm's law : $\mathcal{J} = \sigma \tilde{\mathcal{E}} + \mathcal{J}_e$, σ conductivity
$\mathcal{J}_e$ external current

If $\sigma \to \infty$ but $\mathcal{J}$ remains finite, then we must have $\tilde{\mathcal{E}} \to 0$.

Hence, if material 2 is a perfect conductor, then $\tilde{\mathcal{E}} = 0$ and hence

$$\nu \times \mathcal{E} = 0 \quad \text{on } S.$$

This implies the condition

$$(\mu \mathcal{H}) \cdot \nu = \text{constant} \quad \text{on } S,$$

because

$$\partial_t \left((\mu \mathcal{H}) \cdot \nu \right) = (\mu \partial_t \mathcal{H}) \cdot \nu$$
$$= - \operatorname{curl} \mathcal{E} \cdot \nu$$
$$\overset{\text{exercise}}{=} \operatorname{div} (\underbrace{\nu \times \mathcal{E}}_{=0}) = 0$$

1.4 The wave equation

Consider the linear Maxwell equations in $\mathbb{R}^3$ in a homogeneous, isotropic material ($\rightarrow \varepsilon, \mu \in \mathbb{R}$ constant) without charges and currents ($\rightarrow \rho(t,x) \equiv 0$, $\mathcal{J}(t,x) \equiv 0$):

$$\varepsilon \partial_t \mathcal{E} = \operatorname{curl} \mathcal{H} \qquad \operatorname{div} \mathcal{E} = 0$$

$$\mu \partial_t \mathcal{H} = -\operatorname{curl} \mathcal{E} \qquad \operatorname{div} \mathcal{H} = 0$$

Substitute the second equation into the first one:

$$\partial_t^2 \mathcal{E} = \frac{1}{\varepsilon} \operatorname{curl} \partial_t \mathcal{H} = -\frac{1}{\varepsilon\mu} \operatorname{curl}(\operatorname{curl} \mathcal{E})$$

Since $\operatorname{curl} \mathcal{E} = \begin{pmatrix} \partial_2 \mathcal{E}_3 - \partial_3 \mathcal{E}_2 \\ \partial_3 \mathcal{E}_1 - \partial_1 \mathcal{E}_3 \\ \partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1 \end{pmatrix}$, the first entry of $\operatorname{curl}(\operatorname{curl} \mathcal{E})$ is

$$\partial_2(\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1) - \partial_3(\partial_3 \mathcal{E}_1 - \partial_1 \mathcal{E}_3) + \partial_1^2 \mathcal{E}_1 - \partial_1^2 \mathcal{E}_1$$

$$= \partial_1 \underbrace{(\partial_1 \mathcal{E}_1 + \partial_2 \mathcal{E}_2 + \partial_3 \mathcal{E}_3)}_{= \operatorname{div} \mathcal{E} = 0} - \underbrace{(\partial_1^2 \mathcal{E}_1 + \partial_2^2 \mathcal{E}_1 + \partial_3^2 \mathcal{E}_1)}_{= \Delta \mathcal{E}_1}$$

Similar for the other entries $\Rightarrow \operatorname{curl}(\operatorname{curl} \mathcal{E}) = -\Delta \mathcal{E} = -\begin{pmatrix} \Delta \mathcal{E}_1 \\ \Delta \mathcal{E}_2 \\ \Delta \mathcal{E}_3 \end{pmatrix}$

Hence, we obtain

$$\partial_t^2 \mathcal{E} = c^2 \Delta \mathcal{E}, \qquad \partial_t^2 \mathcal{H} = c^2 \Delta \mathcal{H}$$

with $c := \frac{1}{\sqrt{\varepsilon\mu}}$. In vacuum, c is the speed of light.

Hence, Maxwell's equations are reduced to six decoupled scalar wave equations. Therefore, we will study the wave equation as a simple special case.

(a) Solution on $\mathbb{R}$

Let $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$. Then, the solution of the initial value problem

$$\begin{aligned} \partial_t^2 u(t,x) &= c^2 \partial_x^2 u(t,x) \qquad x \in \mathbb{R},\ t \geq 0 \\ u(0,x) &= u_0(x) \\ \partial_t u(0,x) &= v_0(x) \end{aligned}$$

is given by d'Alembert's formula:

$$u(t,x) = \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\, d\xi$$

Check: $u(0,x) = \frac{1}{2}\left(u_0(x) + u_0(x)\right) = u_0(x)$ ✓

$$\cancel{\partial_t u(0,x) = \frac{1}{2}\Big(\underbrace{c u_0'(x) - c u_0'(x)}_{=0}\Big)}$$

$$\begin{aligned} \partial_t u(t,x) &= \frac{1}{2}\left(c u_0'(x+ct) - c u_0'(x-ct)\right) \\ &\quad + \partial_t\left(\frac{1}{2}\int_0^t v_0(x+cs)\, ds + \frac{1}{2}\int_0^t v_0(x-c\tau)\, d\tau\right) \\ &= \frac{1}{2}\left(c u_0'(x+ct) - c u_0'(x-ct)\right) + \frac{1}{2}\left(v_0(x+ct) + v_0(x-ct)\right) \end{aligned}$$

$\Rightarrow \partial_t u(0,x) = v_0(x)$ ✓

With
$$F(y) := \frac{1}{2}\left(u_0(y) + \frac{1}{c}\int_0^y v_0(\xi)\, d\xi\right)$$
$$G(y) := \frac{1}{2}\left(u_0(y) + \frac{1}{c}\int_y^0 v_0(\xi)\, d\xi\right)$$

we have $u(t,x) = F(x+ct) + G(x-ct)$ and hence