Define $\mathcal{H} \overset{(14b)}{:=} g + \frac{1}{\mu}\operatorname{curl}\mathcal{E} \quad \in H(\operatorname{curl})$

$\Rightarrow \begin{pmatrix}\mathcal{E}\\ \mathcal{H}\end{pmatrix} \in D(M)$.

Same arguments $\Rightarrow$ $\operatorname{rg}(I-M)$ dense in X $\Rightarrow$ (a)

3. Now let $\begin{pmatrix}f\\ g\end{pmatrix} \in X_0$ (instead of X)

$$\left.\begin{aligned} 0 &= \operatorname{div}(\varepsilon f) \overset{(14a)}{=} \operatorname{div}(\varepsilon\mathcal{E}) + \underbrace{\operatorname{div}\operatorname{curl}\mathcal{H}}_{=0 \text{ (distribution)}} \\ 0 &= \operatorname{div}(\mu g) \overset{(14b)}{=} \operatorname{div}(\mu\mathcal{H}) - \underbrace{\operatorname{div}\operatorname{curl}\mathcal{E}}_{=0 \text{ (distribution)}} \end{aligned}\right\} \Rightarrow \begin{pmatrix}\mathcal{E}\\ \mathcal{H}\end{pmatrix} \in D(M)\cap X_0 = D(M_0)$$

Step 2 $\Rightarrow$ M_0 skew-adjoint in X_0 (S)

---

(62)

Prove (c): ~~...~~ If

$$\begin{pmatrix}\mathcal{E}\\ \mathcal{H}\end{pmatrix} \in D(M), \quad \begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\end{pmatrix} := M\begin{pmatrix}\mathcal{E}\\ \mathcal{H}\end{pmatrix} \overset{4}{=} \begin{pmatrix}\frac{1}{\varepsilon}\operatorname{curl}\mathcal{H}\\ -\frac{1}{\mu}\operatorname{curl}\mathcal{E}\end{pmatrix}$$

then

$$\left.\begin{aligned} \operatorname{div}(\varepsilon\mathcal{F}_1) &= \operatorname{div}\operatorname{curl}\mathcal{H} = 0 \\ \operatorname{div}(\mu\mathcal{F}_2) &= -\operatorname{div}\operatorname{curl}\mathcal{E} = 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix}\mathcal{F}_1\\ \mathcal{F}_2\end{pmatrix} \in X_0 \Rightarrow \text{(c)}$$

(distributional sense)

$\blacksquare$

Remarks:

1. Well-posedness of Maxwell's equations (8) - (10) on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ and boundary conditions

$$\mathcal{E}(t) \times \nu = 0 \qquad t \in \mathbb{R},\ x \in \partial\Omega$$
$$\mu\mathcal{H}(t) \cancel{\times} \cdot \nu = 0 \qquad \nu \text{ outer unit normal vector}$$

can be shown in a similar way.
Additional difficulty: Interpretation of the boundary conditions (trace operators), modified integration by parts formulas (11), (12).

2. Well-posedness of Maxwell's equations with nonzero current density J and nonzero charge density ρ (cf. 1.3) can be discussed via the variation-of-constants formula:

If $u \in D(A)$ and f is sufficiently regular, then the classical solution

of the inhomogeneous ACP

$$\dot{u}(t) = Au(t) + f(t) \qquad t \geq 0$$
$$u(0) = u_0$$

is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s)\, ds.$$

2. The Yee scheme

2.1 Finite difference method for the wave equation

Consider the wave equation on $\Omega := (0, 2\pi)$ with periodic boundary conditions:

$$
(1)\quad \begin{cases}
\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x) & t \geq 0,\ x \in \Omega & \text{PDE} \\
u(t,0) = u(t,2\pi) & t \geq 0 & \text{boundary cond.} \\
u(0,x) = u_0(x) & x \in \Omega & \text{initial conditions} \\
\partial_t u(0,x) = v_0(x) & &
\end{cases}
$$

Assume that $u_0(0) = u_0(2\pi)$

Assume that a sufficiently smooth classical solution exists.

Equivalent formulation as a first-order problem in time:

$$
(2)\quad \begin{cases}
\partial_t u(t,x) = v(t,x) & t \geq 0,\ x \in \Omega \\
\partial_t v(t,x) = c^2 \partial_x^2 u(t,x) & \\
u(t,0) = u(t,2\pi) & \\
u(0,x) = u_0(x) & \\
v(0,x) = v_0(x) &
\end{cases}
$$

~~Spatial discretization~~ ~~Choose $m \in \mathbb{N}$, let $x_j := jh$ with $h := \frac{2\pi}{m}$ (mesh width)~~

~~Time discretization~~ ~~Choose step size $\tau > 0$, let $t_n :=$~~

Choose $m \in \mathbb{N}$, define mesh width $h := \frac{2\pi}{m}$, let $x_j := jh$ for $j = 0, \dots, m$

Choose ~~ma~~ step-size $\tau > 0$, let $t_n = n\tau$ for $n = 0, 1, 2, \dots$

Goal: Compute approximations $u_j^n \approx u(t_n, x_j)$

Approximate derivatives by difference quotients:

$$\partial_t^2 u(t_n, x_j) \approx \frac{u(t_n+\tau, x_j) - 2u(t_n, x_j) + u(t_n-\tau, x_j)}{\tau^2}$$

$$\partial_x^2 u(t_n, x_j) \approx \frac{u(t_n, x_{j+1}) - 2u(t_n, x_j) + u(t_n, x_{j-1})}{h^2}$$

This motivates the following finite difference scheme:

$$(3) \qquad \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\tau^2} = c^2 \, \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} \qquad \begin{array}{l} n = 1, 2, \ldots \\ j = 1, \ldots, m-1 \\ u_0^n = u_m^n \end{array}$$

Initial data: $u_j^0 = u_0(x_j)$

Define

$$u^n := \begin{pmatrix} u_1^n \\ \vdots \\ u_m^n \end{pmatrix} \in \mathbb{R}^m, \qquad A := \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & -2 \end{pmatrix}$$

Then, (3) is equivalent to

$$(4) \qquad u^{n+1} - 2u^n + u^{n-1} = \tau^2 c^2 A u^n \qquad n = 1, 2, \ldots$$

This is an explicit two-step method: $u^n, u^{n-1} \longrightarrow u^n$

The first approximation u^1 is obtained by a starting step:

$$u^1 := u^0 + \tau v^0 + \frac{\tau^2 c^2}{2} A u^0, \qquad v^0 := \begin{pmatrix} v_0(x_1) \\ \vdots \\ v_0(x_m) \end{pmatrix}$$

Equivalent one-step formulation:

$$(5) \quad \begin{cases} v^{n+1/2} = v^n + \frac{\tau}{2} c^2 A u^n \\ u^{n+1} = u^n + \tau v^{n+1/2} \\ v^{n+1} = v^{n+1/2} + \frac{\tau}{2} c^2 A u^{n+1} \end{cases}$$

$$v^n \approx \begin{pmatrix} v(t_n, x_1) \\ \vdots \\ v(t_n, x_m) \end{pmatrix}, \quad \text{cf. (2)}$$

(a) Stability

Observation: The scheme (3)/(4)/(5) yields unbounded approximations if the step-size τ is too large. Instability!

Which condition must be fulfilled to obtain a bounded numerical solution?

Ansatz: Fourier transform

For every vector $u^n \in \mathbb{R}^m$, there is a unique representation

$$u_j^n = \sum_{k=1}^{m} e^{ikx_j}\, \hat{u}_k^n \qquad (\hat{u}_k^n \in \mathbb{R})$$

Substitute into (3):

$$\frac{1}{\tau^2}\left(u_j^{n+1} - 2u_j^n + u_j^{n-1}\right) = \sum_{k=1}^{m} e^{ikx_j}\, \frac{1}{\tau^2}\left(\hat{u}_k^{n+1} - 2\hat{u}_k^n + \hat{u}_k^{n-1}\right)$$

$$\overset{(3)}{=} \frac{c^2}{h^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) = \frac{c^2}{h^2}\sum_{k=1}^{m} e^{ikx_j}\, \hat{u}_k^n \left(e^{ikh} - 2 + e^{-ikh}\right)$$

Compare coefficients:

$$\hat{u}_k^{n+1} - 2\hat{u}_k^n + \hat{u}_k^{n-1} = \left(\frac{\tau c}{h}\right)^2 \hat{u}_k^n \underbrace{\left(e^{ikh} - 2 + e^{-ikh}\right)}_{= 2(\cos(kh) - 1)}$$

Advantage: Decoupling with respect to k.

Equivalent formulation:

$$\begin{pmatrix} \hat{u}_k^{n+1} \\ \hat{u}_k^n \end{pmatrix} = \begin{pmatrix} 2\alpha & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_k^n \\ \hat{u}_k^{n-1} \end{pmatrix} \qquad \alpha := \left(\frac{\tau c}{h}\right)^2 (\cos(kh) - 1)$$

The ~~[illegible]~~ $\hat{u}_k^n$ and hence the u_j^n stay bounded for all n if and only if the eigenvalues λ of $\begin{pmatrix} 2\alpha & -1 \\ 1 & 0 \end{pmatrix}$ satisfy $|\lambda| \le 1$.

$$\lambda = \alpha \pm \sqrt{\alpha^2 - 1}$$