Case I: $\alpha^2-1>0$

If $\alpha>1$, then $\lambda=\alpha+\sqrt{\alpha^2-1}>1$ $\Rightarrow$ instability

If $\alpha<-1$, then $\lambda=\alpha-\sqrt{\alpha^2-1}<-1$ $\Rightarrow$ instability

Case II: $\alpha^2-1\leq 0$

$\Rightarrow$ $\sqrt{\alpha^2-1}=\pm i\sqrt{1-\alpha^2}$

$\Rightarrow$ $|\lambda|^2=\alpha^2+\sqrt{1-\alpha^2}^2=1$

$\Rightarrow$ stability

Hence, the scheme (3)/(4)/(5) is stable if and only if

$0\geq\alpha^2-1$, i.e. $\alpha\in[-1,1]$

$$\Leftrightarrow\quad -1\leq\left(\frac{\tau c}{h}\right)^2\left(\cos(kh)-1\right)+1\leq 1$$

always true

$$\Leftrightarrow\quad 2\geq\left(\frac{\tau c}{h}\right)^2\left(1-\cos(kh)\right)$$

This must be true for all $k=1,\dots,m$ (if $\hat{u}_k^n\neq 0$)

For $k=\frac{m}{2}$ we have $1-\cos\left(\frac{mh}{2}\right)=1-\cos(\pi)=2$

(assume m even)

This yields the stability condition

$$\tau\leq\frac{h}{c}$$

(CFL condition, Courant, Friedrichs, Lewy 1928)

A better resolution in space requires a smaller step-size in time!

(b) Accuracy

If the exact solution of (1)/(2) is sufficiently smooth and the stability condition $\tau \leq \frac{h}{c}$ is fulfilled, then the error of the scheme (3)/(4)/(5) is bounded by

$$\max_{n=0,\dots,N} \max_{j=1,\dots,m} |u(t_n,x_j) - u_j^n| \leq C\cdot(\tau^2 + h^2) \leq \underbrace{C\cdot\left(\frac{1}{c^2}+1\right)}_{=:\tilde{C}} h^2$$

(order 2 in space and time)

Proof: Exercise

In special cases, the numerical approximation is even <u>exact</u>:

<u>Lemma 2.1</u>

Let $\tau = \frac{h}{c}$ and assume that the starting values of (3)/(4)

$$u_j^0 = u(0,x_j) = u_0(x_j), \quad u_j^1 = u(t_1,x_j) \quad \forall j = 1,\dots,m$$

are exact (instead of approximation $u_j^1 \approx u(t_1,x_j)$ by starting step).
Then, (3)/(4) yields the exact solution for all $n \in \mathbb{N}$.

Remark: The choice $\tau = \frac{h}{c}$ is called the <u>magic time-step</u>.

Proof: Let $\tilde{u}_0$ and $\tilde{v}_0$ be the periodic continuation of u_0 and v_0, i.e.

$$\left.\begin{aligned} \tilde{u}_0(x + \ell\cdot 2\pi) &:= u_0(x) \\ \tilde{v}_0(x + \ell\cdot 2\pi) &:= v_0(x) \end{aligned}\right\} \quad \forall x \in [0,2\pi],\ \ell \in \mathbb{Z}$$

According to 1.4 (a), the exact solution of (1)/(2) is

$$u(t,x) = F(x+ct) + G(x-ct)$$

with $$F(y) = \frac{1}{2}\left(\tilde{u}_0(y) + \frac{1}{c}\int_0^y \tilde{v}_0(\xi)\,d\xi\right)$$

$$G(y) = \frac{1}{2}\left(\tilde{u}_0(y) + \frac{1}{c}\int_y^0 \tilde{v}_0(\xi)\,d\xi\right).$$

(The function $u(t,\cdot)$ is defined on $\mathbb{R}$ but can be restricted to $[0,2\pi]$.)

$$(3) \overset{\tau=\frac{h}{c}}{\iff} u_j^{n+1} - 2u_j^n + u_j^{n-1} = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

$$\iff u_j^{n+1} = u_{j+1}^n + u_{j-1}^n - u_j^{n-1}$$

Induction: Assume that $u_j^k = u(t_k, x_j)$ is exact for all $k=0,\dots,n$ and $j=1,\dots,m$:

$$u_j^k = F(x_j + ct_k) + G(x_j - ct_k) \overset{ct_k = k\cdot h}{=} F((j+k)h) + G((j-k)h)$$

Hence:

$$u_j^{n+1} = \underbrace{F((j+n+1)h) + G((j-n+1)h)}_{u_{j+1}^n} + \underbrace{F((n+j-1)h) + G((j-1-n)h)}_{u_{j-1}^n}$$

$$\underbrace{-\ F((j+n-1)h) - G((j-n+1)h)}_{=-u_j^{n-1}}$$

$$= F((j+n+1)h) + G((j-(n+1))h).$$

$\blacksquare$

(c) Dispersion

The harmonic wave

$$u(t,x) = \exp(i\omega t - ikx) \in \mathbb{C}$$

solves (1)/(2) with initial data $u_0(x) = \exp(-ikx)$ and $v_0(x) = i\omega \exp(-ikx)$ if and only if the dispersion relation

$$\omega = \pm ck$$

holds.

Proof: Exercise

If $\tau = \frac{h}{c}$ (magic time-step), then the numerical approximation given by (3)/(4) is exact.

If $\tau < \frac{h}{c}$ and $n \gg 1$ is large (many time-steps), then the numerical solution still looks like a harmonic wave, but there is a shift between the numerical and the exact solution due to <u>numerical dispersion</u>.

Substitute $u_j^n := \exp(i\omega t_n - i\tilde{k} x_j)$ with unknown $\tilde{k}$ into (3) and use $t_{n\pm1} = t_n \pm \tau$, $x_{j\pm1} = x_j \pm h$:

$$\exp(i\omega t_n - i\tilde{k}x_j)\frac{1}{\tau^2}\Big(\exp(i\omega\tau) - 2 + \exp(-i\omega\tau)\Big)$$
$$\overset{!}{=} \exp(i\omega t_n - i\tilde{k}x_j)\frac{c^2}{h^2}\Big(\exp(i\tilde{k}h) - 2 + \exp(-i\tilde{k}h)\Big)$$

With $\exp(i\beta) + \exp(-i\beta) = 2\cos(\beta)$, this yields the

<u>numerical dispersion relation</u>

$$\frac{\cos(\omega\tau) - 1}{\tau^2} \overset{!}{=} c^2\,\frac{\cos(\tilde{k}h) - 1}{h^2}$$

If $\tau = \frac{h}{c}$ (magic time-step), then $\tilde{k} \overset{!}{=} \frac{\omega\tau}{h} = \pm k$ is exact.
$\omega = \pm ck = \pm\frac{h}{\tau}k$

Now let $\tau < \frac{h}{c}$ and keep $r := \frac{h}{\tau c} > 1$ fixed.
Since $\cos(\varepsilon) = 1 - \frac{\varepsilon^2}{2} + O(\varepsilon^4)$ for $\varepsilon \to 0$, we find for $\tau \to 0$

$$\frac{1}{\tau^2}\left(-\frac{(\omega\tau)^2}{2} + O\big((\omega\tau)^4\big)\right) = \frac{c^2}{h^2}\left(-\frac{(\tilde{k}h)^2}{2} + O\big((\tilde{k}h)^4\big)\right)$$
$$\Longleftrightarrow\quad -\frac{\omega^2}{2} + O(\omega^4\tau^2) = -\frac{c^2}{2}\tilde{k}^2 + O(c^2\tilde{k}^4\underbrace{h^2}_{=(r\tau c)^2})$$
$$\Longleftrightarrow\quad \tilde{k}^2 = \frac{\omega^2}{c^2} + O(\tau^2)$$