(d) Error of the full discretization

Applying (b) and (c) with

$$q(t) = E[t, \dots] \qquad A = \frac{1}{\varepsilon} \widehat{\mathrm{curl}} \qquad \omega = 0$$

$$v(t) = H[t, \dots] \qquad B = -\frac{1}{\mu} \widehat{\mathrm{curl}} \qquad C_\omega = 1$$

yields the error bound

$$\left\| \begin{pmatrix} \Gamma_E \mathcal{E}(t_n, \dots) \\ \Gamma_H \mathcal{H}(t_n, \dots) \end{pmatrix} - \begin{pmatrix} E^n[\dots] \\ H^n[\dots] \end{pmatrix} \right\|_{m, \varepsilon, \mu}$$

$$\le \underbrace{\left\| \begin{pmatrix} \Gamma_E \mathcal{E}(t_n, \dots) \\ \Gamma_H \mathcal{H}(t_n, \dots) \end{pmatrix} - \begin{pmatrix} E[t_n, \dots] \\ H[t_n, \dots] \end{pmatrix} \right\|_{m, \varepsilon, \mu}}_{\le C h^2 \cdot t} + \underbrace{\left\| \begin{pmatrix} E[t_n, \dots] \\ H[t_n, \dots] \end{pmatrix} - \begin{pmatrix} E^n[\dots] \\ H^n[\dots] \end{pmatrix} \right\|_{m, \varepsilon, \mu}}_{C_* \tau^2 \left(\left\| \mathcal{M}^3 \binom{E^0}{H^0} \right\| + \max\limits_{j=0, \dots, n} \dots \right.}$$

under the assumptions

- that $\mathcal{E}(t, \cdot) \in C^3(\bar{\Omega}, \mathbb{R}^3)$, $\mathcal{H}(t, \cdot) \in C^3(\bar{\Omega}, \mathbb{R}^3)$
- that $\varepsilon, \mu : \bar{\Omega} \longrightarrow \mathbb{R}$ are sufficiently smooth, bounded and $\varepsilon(x), \mu(x) \ge \delta$
- that the stability condition (21) holds
  ($\Rightarrow$ CFL condition $\tau \le s h$ for some $s > 0$)

Note that $\left\| \mathcal{M}^3 \binom{E^0}{H^0} \right\|_{m, \varepsilon, \mu} \longrightarrow \left\| \mathcal{M}^3 \binom{\mathcal{E}^0}{\mathcal{H}^0} \right\|_{L^2} \cdot C$ for $h \to 0$

$$\mathcal{M} := \begin{pmatrix} 0 & \frac{1}{\varepsilon} \mathrm{curl} \\ -\frac{1}{\mu} \mathrm{curl} & 0 \end{pmatrix}$$

According to Lemma 2.4, 2.5, the error of the space discretization depends on $\partial_\alpha \partial_\beta \partial_\gamma \mathcal{E}(t, \cdot)$ and $\partial_\alpha \partial_\beta \partial_\gamma \mathcal{H}(t, \cdot)$, too.

3. The Namiki-Zheng-Chen-Zhang method

Linear Maxwell equations on $\Omega = (0,L)^3$ or $\Omega = \mathbb{R}^3$:

(1a)	$\varepsilon \partial_t \mathcal{E} = \operatorname{curl} \mathcal{H}$	in Ω	(3a)	$\mathcal{E} \times \nu = 0$	on $\partial\Omega$
(1b)	$\mu \partial_t \mathcal{H} = -\operatorname{curl} \mathcal{E}$	in Ω	(3b)	$(\mu\mathcal{H}) \cdot \nu = 0$	on $\partial\Omega$
(2a)	$\operatorname{div}(\varepsilon \mathcal{E}) = 0$	in Ω	(4a)	$\mathcal{E}(0,x) = \mathcal{E}^0(x)$	in Ω
(2b)	$\operatorname{div}(\mu \mathcal{H}) = 0$	in Ω	(4b)	$\mathcal{H}(0,x) = \mathcal{H}^0(x)$	in Ω

(cf. (6)-(9) in 2.2, $\mathcal{J} \equiv 0$, $g \equiv 0$)
Conditions (3a), (3b) are omitted if $\Omega = \mathbb{R}^3$

Assumption (A1): $\varepsilon, \mu \in C^\infty(\Omega)$, $\varepsilon(x), \mu(x) \geq \delta$ for some $\delta > 0$.

2

Disadvantage of the Yee scheme: Stability condition $\tau \leq s h$ $(s>0)$
Accurate space discretization $\Rightarrow$ small h $\Rightarrow$ small τ $\Rightarrow$ many time-steps
$\Rightarrow$ high numerical costs

Goal: Find a numerical method of order 2 (or higher) which is unconditionally stable and numerically efficient.

3.1 The Crank-Nicolson method

Consider the initial-value problem

$$\dot{y}(t) = Ay(t) + f(t) \qquad t \in [0, t_{end}]$$
$$y(0) = y_0$$

with $y(t), y_0 \in \mathbb{R}^d$, $f: [0, t_{end}] \longrightarrow \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$.

Typically, A is the discretization of a differential operator.

Approximate $y^n \approx y(t_n)$ with the implicit midpoint rule:

$$y^{n+1} = y^n + \frac{\tau}{2} A(y^{n+1} + y^n) + \tau f(t_n + \tfrac{\tau}{2})$$

Equivalent:

$$(I - \tfrac{\tau}{2} A) y^{n+1} = (I + \tfrac{\tau}{2} A) y^n + \tau f(t_n + \tfrac{\tau}{2})$$

In each time-step, a linear system with a $(d \times d)$-matrix has to be solved.

Example: Heat equation on $(0, L)^2$, space discretization with finite differences, $m \in \mathbb{N}$ points in each direction $\Rightarrow d = O(m^2)$

Assume that $A = U \Lambda U^*$, U unitary, $\Lambda = \mathrm{diag}(\lambda_k)$, $\mathrm{Re}(\lambda_k) \leq 0$

If $f(t) \equiv 0$, then the Crank-Nicolson method is unconditionally stable:

$$\|y^n\|_2 \leq \|y^0\| \qquad \text{for all } n \in \mathbb{N} \text{ and arbitrary } \tau > 0$$

Proof: exercise

Problem: If Crank-Nicolson is applied to the semi-discretized Maxwell equations

$$\begin{pmatrix}\partial_t E\\ \partial_t H\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\widehat{\mathrm{curl}} \\ -\frac{1}{\mu}\widehat{\mathrm{curl}} & 0\end{pmatrix}\begin{pmatrix}E\\ H\end{pmatrix}$$

then $\binom{E}{H} \mathrel{\hat=} y \in \mathbb{R}^d$, $\hat{A} \mathrel{\hat=} \tilde{M} \in \mathbb{R}^{d\times d}$ with $d = \mathcal{O}(6n^3)$.

Solving a linear system with Gauß ~~[illegible]~~ elimination or Cholesky decomposition requires $\mathcal{O}(d^3)$ operations.

Example: $n = 100 \Rightarrow d = 6\cdot 10^6 \Rightarrow d^3 = 6^3\cdot 10^{18}$

Too expensive!

Same problem if the implicit midpoint rule is replaced by any other A-stable Runge-Kutta method.

3.2 Derivation of the NZCZ method

Goal: Semi-discretization in time, i.e. compute

$$\mathcal{E}^n(x) \approx \mathcal{E}(t_n, x), \qquad \mathcal{H}^n(x) \approx \mathcal{H}(t_n, x)$$

Notation: $\mathcal{E}(t) = \mathcal{E}(t,x)$ etc., as in 1.5.

For simplicity, we consider the case $\Omega = \mathbb{R}^3$, i.e. without (3).

Idea of Namiki (1999, 2000), Zheng, Chen, Zhang (2000):
Split the curl operator into two parts.

$$\mathrm{curl}\,\mathcal{F} = \underbrace{\begin{pmatrix}\partial_2 \mathcal{F}_3\\ \partial_3 \mathcal{F}_1\\ \partial_1 \mathcal{F}_2\end{pmatrix}}_{=:\,C_1\mathcal{F}} - \underbrace{\begin{pmatrix}\partial_3 \mathcal{F}_2\\ \partial_1 \mathcal{F}_3\\ \partial_2 \mathcal{F}_1\end{pmatrix}}_{=:\,C_2\mathcal{F}}$$

Define

$$A = \begin{pmatrix} 0 & \frac{1}{\varepsilon} C_1 \\ \frac{1}{\mu} C_2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\frac{1}{\varepsilon} C_2 \\ -\frac{1}{\mu} C_1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad A + B = \begin{pmatrix} 0 & \frac{1}{\varepsilon}\,\mathrm{curl} \\ -\frac{1}{\mu}\,\mathrm{curl} & 0 \end{pmatrix} =: M$$ Maxwell operator, cf. 1.6

Spaces: $X, X_0, D(M), D(M_0)$ etc. as in 1.6

$$D(A) := \left\{ \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} \in X : \begin{pmatrix} C_1 \mathcal{G} \\ C_2 \mathcal{F} \end{pmatrix} \in X \right\}$$

$$D(B) := \left\{ \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} \in X : \begin{pmatrix} C_2 \mathcal{G} \\ C_1 \mathcal{F} \end{pmatrix} \in X \right\}$$

$$(1) \Leftrightarrow \partial_t u = M u = Au + Bu \quad \text{for } u(t,x) = \begin{pmatrix} \mathcal{E}(t,x) \\ \mathcal{H}(t,x) \end{pmatrix} \in D(A) \cap D(B) \subseteq D(M)$$

---

(8)

NZCZ method (Namiki, Zheng, Chen, Zhang):

$$u^{n+1} = \Phi_\tau u^n = \underbrace{\left(I - \tfrac{\tau}{2} B\right)^{-1}}_{\text{impl.}} \underbrace{\left(I + \tfrac{\tau}{2} A\right)}_{\text{expl.}} \underbrace{\left(I - \tfrac{\tau}{2} A\right)^{-1}}_{\text{impl.}} \underbrace{\left(I + \tfrac{\tau}{2} B\right)}_{\text{expl.}} u^n$$

$$u^n(x) = \begin{pmatrix} \mathcal{E}^n(x) \\ \mathcal{H}^n(x) \end{pmatrix}$$

Remark: This is a splitting method. In each sub-step, we ignore either A or B and approximate with one step of the explicit or implicit Euler method with step-size $\frac{\tau}{2}$.

⌜ ODE $\dot{y} = f(y)$

Explicit Euler method: $y^{n+1} = y^n + \tau f(y^n)$

Implicit Euler method: $y^{n+1} = y^n + \tau f(y^{n+1})$ ⌟