- Local error

Let $w \in D(M_0^3) = D(M^3) \cap X_0$

$$\Rightarrow \quad M_0^k \Lambda_j(\tau) w \in \underbrace{D(M_0^{3-k})}_{\substack{\subseteq H^2(\Omega)^6 \\ \text{Lemma 3.3}}} \subseteq D(AB) \cap D(A) \quad \text{for } k=0,1, \; j \in \mathbb{N}_0$$

2nd order differential operator

$$\Phi_\tau w - T_0(\tau) w$$

$$= (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1}\Big[(I + \tfrac{\tau}{2}A)(I + \tfrac{\tau}{2}B) - (I - \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}B)T_0(\tau)\Big] w$$

$$= (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1}\Big[I + \tfrac{\tau}{2}\underbrace{(A+B)}_{=M_0} + \tfrac{\tau^2}{4}AB - \Big(I - \tfrac{\tau}{2}\underbrace{(A+B)}_{=M_0} + \tfrac{\tau^2}{4}AB\Big)T_0(\tau)\Big] w$$

$$(9) \quad \underset{\substack{\uparrow \\ \text{exercise}}}{\overset{(8)}{=}} \tau^3 (I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1}\Big[\big(\tfrac{1}{2}\Lambda_2(\tau) - \Lambda_3(\tau)\big)M_0^3 - \tfrac{1}{4}ABM_0\Lambda_1(\tau)\Big] w$$

- Global error

$$u^n - u(t_n) = \Phi_\tau^n u^0 - \underbrace{T(t_n)}_{=T(\tau)^n} u^0$$

$$= \sum_{j=0}^{n-1} \Phi_\tau^{n-j-1}\big(\Phi_\tau - T_0(\tau)\big)u(t_j) \qquad \text{(cf. exercise 10 and proof of Lemma 2.5)}$$

$$\overset{(9)}{=} \tau^3 \sum_{j=0}^{n-1} \Phi_\tau^{n-j-1}(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1}\Big[\tfrac{1}{2}\Lambda_2(\tau) - \Lambda_3(\tau)\Big]M_0^3 u(t_j)$$

$$(10) \quad - \frac{\tau^3}{4}\sum_{j=0}^{n-1}\Phi_\tau^{n-j-1}(I - \tfrac{\tau}{2}B)^{-1}(I - \tfrac{\tau}{2}A)^{-1}AB\underbrace{(I - M_0)^{-2}(I - M_0)^2}_{=I}M_0\Lambda_1(\tau)u(t_j)$$

- For every $v \in X_0$:

$$\|AB(I - M_0)^{-2}v\|_{L^2} \le \|(I - M_0)^{-2}v\|_{H^2} \qquad \text{(because } AB \text{ is a 2nd order differential operator)}$$

$$\le C \cdot \|(I - M_0)^{-2}v\|_{D(M_0^2)} \qquad \text{(Lemma 3.3)}$$

$$\|(I-M_0)^{-2}v\|_{D(M_0^2)} = \|(I-M_0)^{-2}v\|_X + \|\underbrace{M_0^2}_{=(I-M_0)^2+2(M_0-I)+I}(I-M_0)^{-2}v\|_X$$

$$\leq 2\,\|(I-M_0)^{-2}v\|_X + \|v\|_X + 2\,\|(I-M_0)^{-1}v\|_X$$

$$\leq C\,\|v\|_X \quad \text{because } \|(I-M_0)^{-1}\|_X \leq 1$$

$\Rightarrow$ $AB(I-M_0)^{-2} : X_0 \to X_0$ is bounded.

- Use equivalence $\|\cdot\|_X \sim \|\cdot\|_{L^2}$ and boundedness of Φ_τ^{n-j-1}, $(I-\frac{\tau}{2}B)^{-1}$, $(I-\frac{\tau}{2}A)^{-1}$, $\Lambda_j(\tau)$:

" $$\|u^n - u(t_n)\|_{L^2(\Omega)} \underset{(20)}{\leq} C\tau^3 n \max_{j=0,\dots,n-1} \|M_0^3 u(t_j)\|_{L^2(\Omega)}$$

$$+\, C\tau^3 \underbrace{n}_{\max_j} \underbrace{\|AB(I-M_0)^{-2}\|_X}_{\leq C} \|(I-M_0)^2 M_0 \Lambda_1(\tau) u(t_j)\|_{L^2}$$

$\tau n = t_n \leq t_{end}$

$\|M_0 u(t_j)\|_{L^2} = \|M_0 T(t_j) u^0\|_{L^2} \leq C\|M_0 u^0\|$, $\quad \|M_0^\ell u^0\|_{L^2} \leq C\left(\|u^0\|_{L^2} + \|M_0^3 u^0\|_{L^2}\right)$ for $\ell = 0,1,2$ ■

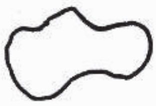
Remark:

A similar error analysis can be carried out for Maxwell's equations (1), (2), (4) on the cube $\Omega = (0,L)^3$ with boundary conditions (3).

Additional technical difficulties due to boundary conditions.

4. Finite element methods

So far, we have assumed that $\Omega = \mathbb{R}^3$ or $\Omega = (0,L)^3$. In most applications, this is not true.

If $\Omega =$  is not a parallelpiped, then finite difference methods or Yee grids are not suitable.

Better method for space discretization?

4.1 Weak formulation

Problem setting

(1a) $\varepsilon \partial_t \mathcal{E} = \operatorname{curl} \mathcal{H} - \mathcal{J}$ in Ω

(1b) $\mu \partial_t \mathcal{H} = -\operatorname{curl} \cancel{\mathcal{H}}\, \mathcal{E}$

(2a) $\operatorname{div}(\varepsilon \mathcal{E}) = \rho$ in Ω

(2b) $\operatorname{div}(\mu \mathcal{H}) = 0$

(3a) $\mathcal{E} \times \nu = 0$ on Γ

(3b) $(\mu \mathcal{H}) \cdot \nu = 0$

(4a) $\mathcal{E}(0,\cdot) = \mathcal{E}^0$ in Ω

(4b) $\mathcal{H}(0,\cdot) = \mathcal{H}^0$

Assumptions:

- $\Omega \subseteq \mathbb{R}^3$ bounded, simply connected polyhedron with connected Lipschitz continuous boundary Γ and unit outward normal $\nu : \Gamma \to \mathbb{R}^3$.
  Cf. Definition 3.1, p. 38 in Monk's book

- $\varepsilon, \mu \in L^\infty(\Omega)$ and $\delta \leq \varepsilon$, $\delta \leq \mu$ for some $\delta > 0$.
- Compatibility condition:

$$\partial_t \rho + \operatorname{div} \mathcal{J} = 0 \qquad (5)$$

- There is a unique solution $\mathcal{E}, \mathcal{H}$ such that

$$\mathcal{E} \in C^1\left([0,t_{end}], L^2(\Omega)^3\right) \cap C^0\left([0,t_{end}], H_0(\operatorname{curl})\right)$$
$$\mathcal{H} \in C^1\left([0,t_{end}], L^2(\Omega)^3\right) \cap C^0\left([0,t_{end}], H(\operatorname{curl})\right)$$

where
$$H(\operatorname{curl}) = \left\{ \mathcal{F} \in L^2(\Omega)^3 : \operatorname{curl} \mathcal{F} \in L^2(\Omega)^3 \right\}$$
$$H_0(\operatorname{curl}) = \left\{ \mathcal{F} \in H(\operatorname{curl}) : \mathcal{F} \times \nu = 0 \text{ on } \Gamma \right\}$$

cf. Corollary 1.10 and Remarks at the end of Ch. 1.

The boundary conditions are to be understood in the sense of traces in $(H^{-1/2}(\Gamma))^3$ and $H^{-1/2}(\Gamma)$. See ch. 3.5.2 and 3.5.3 in Monk's book.

Integration by parts:

(6) $$\int_\Omega \mathcal{F} \cdot \operatorname{curl} \mathcal{G} \, dx = \int_\Omega \mathcal{G} \cdot \operatorname{curl} \mathcal{F} \, dx \qquad \forall \mathcal{F} \in H_0(\operatorname{curl}),\ \mathcal{G} \in H(\operatorname{curl})$$

(7) $$\int_\Omega \mathcal{F} \cdot \nabla \varphi \, dx = -\int_\Omega \varphi \operatorname{div} \mathcal{F} \, dx \qquad \forall \mathcal{F} \in H(\operatorname{div}),\ \mathcal{F} \cdot \nu|_\Gamma = 0,\ \varphi \in H^1(\Omega)$$

Cf. (11), (12) in ch. 1.6

Multiply (1a) with $\phi \in H_0(\operatorname{curl})$ and integrate:

$$\int_\Omega \varepsilon (\partial_t \mathcal{E}) \cdot \phi \, dx = \int_\Omega (\operatorname{curl} \mathcal{H}) \cdot \phi \, dx - \int_\Omega \mathcal{J} \phi \, dx$$
$$\overset{(6)}{=} \int_\Omega \mathcal{H} \cdot \operatorname{curl} \phi \, dx - \int_\Omega \mathcal{J} \phi \, dx$$

Multiply (1b) with $\psi \in L^2(\Omega)^3$ and integrate

$$\int_\Omega \mu (\partial_t \mathcal{H}) \cdot \psi \, dx = -\int_\Omega (\operatorname{curl} \mathcal{E}) \cdot \psi \, dx$$

Define $\langle \mathcal{F}, \mathcal{G} \rangle := \int_\Omega \mathcal{F}(x) \cdot \mathcal{G}(x)\, dx$ for $\mathcal{F}, \mathcal{G} \in L^2(\Omega)^3$.

Weak formulation

Find $\mathcal{E}(t,\cdot) \in H_0(\mathrm{curl})$, $\mathcal{H}(t,\cdot) \in L^2(\Omega)^3$ such that

(8) $\langle \varepsilon \partial_t \mathcal{E}, \phi \rangle = \langle \mathcal{H}, \mathrm{curl}\, \phi \rangle - \langle \mathcal{J}, \phi \rangle \quad \forall \phi \in H_0(\mathrm{curl})$

(9) $\langle \mu \partial_t \mathcal{H}, \psi \rangle = - \langle \mathrm{curl}\, \mathcal{E}, \psi \rangle \quad \forall \psi \in L^2(\Omega)^3$

for all $t \in [0, t_{end}]$ with initial conditions (4a), (4b).

4.2 Galerkin ansatz

Choose finite-dimensional subspaces $U_h \subseteq H_0(\mathrm{curl})$, $V_h \subseteq L^2(\Omega)^3$.
Find $\mathcal{E}^h(t,\cdot) \in U_h$, $\mathcal{H}^h(t,\cdot) \in V_h$ such that

(10) $\langle \varepsilon \partial_t \mathcal{E}^h, \phi^h \rangle = \langle \mathcal{H}^h, \mathrm{curl}\, \phi^h \rangle - \langle \mathcal{J}, \phi^h \rangle \quad \forall \phi^h \in U_h$

(11) $\langle \mu \partial_t \mathcal{H}^h, \psi^h \rangle = - \langle \mathrm{curl}\, \mathcal{E}^h, \psi^h \rangle \quad \forall \psi^h \in V_h$

for all $t \in [0, t_{end}]$ with initial conditions

(12a) $\mathcal{E}^h(0,\cdot) = P^\varepsilon \mathcal{E}^0$

(12b) $\mathcal{H}^h(0,\cdot) = P^H(\mathcal{H}^0)$

where $P^\varepsilon : H_0(\mathrm{curl}) \longrightarrow U_h$ and $P^H : L^2(\Omega)^3 \longrightarrow V_h$ are suitable mappings.