Let $u_1, \dots, u_{d_1}$ be a basis of U_h, and let $v_1, \dots, v_{d_2}$ be a basis of V_h.
Then, there are representations

$$\mathcal{E}^h(t,x) = \sum_{j=1}^{d_1} E_j^h(t) u_j(x) \qquad u_j : \Omega \to \mathbb{R}^3, \quad E_j^h(t) \in \mathbb{R}$$

$$\mathcal{H}^h(t,x) = \sum_{l=1}^{d_2} H_l^h(t) v_l(x) \qquad v_l : \Omega \to \mathbb{R}^3, \quad H_l^h(t) \in \mathbb{R}$$

Choosing $\phi^h := u_i$, $\psi^h := v_k$ in (10) and (11) yields

$$\sum_{j=1}^{d_1} \dot{E}_j^h(t) \langle \varepsilon u_j, u_i \rangle = \sum_{l=1}^{d_2} H_l^h(t) \langle v_l, \operatorname{curl} u_i \rangle - \langle \mathcal{J}, u_i \rangle \qquad \forall i = 1, \dots, d_1$$

$$\sum_{l=1}^{d_2} \dot{H}_j^h(t) \langle \mu v_l, v_k \rangle = - \sum_{j=1}^{d_1} E_j^h(t) \langle \operatorname{curl} u_j, v_k \rangle \qquad \forall k = 1, \dots, d_2$$

Equivalent:

$$M^U \dot{E}^h(t) = A H^h(t) - b$$

$$M^V \dot{H}^h(t) = - A^T E^h(t)$$

$$E^h(t) = \left(E_1^h(t), \dots, E_{d_1}^h(t) \right)^T$$

$$H^h(t) = \left(H_1^h(t), \dots, H_{d_2}^h(t) \right)^T$$

$A = (A_{il})_{i,l} \in \mathbb{R}^{d_1 \times d_2}, \qquad A_{il} = \langle \operatorname{curl} u_i, v_l \rangle$

$M^U = (M^U_{ij})_{i,j} \in \mathbb{R}^{d_1 \times d_1}, \qquad M^U_{ij} = \langle u_i, \varepsilon u_j \rangle$

$M^V = (M^V_{kl})_{k,l} \in \mathbb{R}^{d_2 \times d_2}, \qquad M^V_{kl} = \langle v_k, \mu v_l \rangle$

$b \in \mathbb{R}^{d_1}, \quad b_k = \langle \mathcal{J}, u_k \rangle$

It is easy to show that the mass matrices M^U and M^V are symmetric and positive definite (exercise). Hence, we obtain the ODE

$$\dot{E}^h(t) = (M^U)^{-1} A H^h(t) - (M^U)^{-1} b$$
$$\dot{H}^h(t) = -(M^U)^{-1} A^T E^h(t)$$

with initial data $E^h(0)$, $H^h(0)$ chosen such that

$$\mathcal{E}^h(0,x) = \sum_{j=1}^{d_1} E_j^h(0)\, u_j(x)$$
$$\mathcal{H}^h(0,x) = \sum_{\ell=1}^{d_2} H_\ell^h(0)\, v_\ell(x)$$

The solution can be approximated, e.g., with the Störmer-Verlet method.

4.3 Space discretization by linear Nédélec elements

How to choose V_h and U_h?

(a) Mesh

Since Ω is a polyhedron, we can find a family T_h of sets („elements") $K \subseteq \Omega$ with the following properties.

(i) Every $K \in T_h$ is a tetrahedron excluding the faces and K has non-zero volume.

(ii) $\overline{\Omega} = \bigcup_{K \in T_h} \overline{K}$

(iii) $K_1 \cap K_2 = \emptyset$ for all $K_1, K_2 \in T_h$ with $K_1 \neq K_2$.

(11)

(iv) If ~~$K_1 \neq K_2$~~ $K_1 \neq K_2$ but $\overline{K_1} \cap \overline{K_2} \neq \emptyset$, then $\overline{K_1} \cap \overline{K_2}$ is either a common vertex or a common edge or a common face of K_1 and K_2.

For $K \in \mathcal{T}_h$ let h_K be the diameter of the smallest sphere containing $\overline{K}$.

Define $h := \max_{K \in \mathcal{T}_h} h_K$

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(12)

(6) Degrees of freedom and local basis functions

Let $K \in \mathcal{T}_h$ be a tetrahedron with vertices $\xi_j \in \mathbb{R}^3$ and edges $e_{jk} = [\xi_j, \xi_k]$ for $j,k \in \{1,2,3,4\}$, $j<k$.

Define $R := \left\{ p : K \to \mathbb{R}^3 : \; p(x) = a + b \times x \text{ for } a,b \in \mathbb{R}^3 \right\}$

Every $p \in R$ is a vector of linear polynomials, and the parameters $a,b \in \mathbb{R}^3$ are uniquely determined by the degrees of freedom

$$\Sigma_K := \left\{ \int_{e_{jk}} p \cdot \tau_{jk} \, dl \; , \quad \tau_{jk} \text{ unit vector in the direction of } e_{jk}, \; j,k \in \{1,2,3,4\}, j<k \right\}$$

(six edges, six parameters)

Suitable basis of R?

Let $\lambda_j : K \to \mathbb{R}$ be the barycentric coordinate function of ξ_j, i.e. $\lambda_j(x) = \beta_0^j + \beta^j \cdot x$ is a linear polynomial with $\beta_0^j \in \mathbb{R}$, $\beta^j \in \mathbb{R}^3$ such that

$$\lambda_j(\xi_k) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{else} \end{cases}$$

Define $\phi_{\ell m}(x) := \lambda_\ell(x) \nabla \lambda_m - \lambda_m(x) \nabla \lambda_\ell$

Then, it can be shown that $\{ \phi_{\ell m},\ \ell < m \in \{1,2,3,4\} \}$ is a basis of R, and

$$\int_{e_{jk}} \phi_{\ell m} \cdot \tau_{jk} \, d\ell = \begin{cases} 1 & \text{if } \ell = j, m = k \\ 0 & \text{else} \end{cases}$$

Consequence: For every $p \in R$, we have

$$p(x) = \sum_{j=1}^{3} \sum_{k=j+1}^{4} a_{jk} \phi_{jk}(x) \quad \text{with } a_{jk} = \int_{e_{jk}} p \cdot \tau_{jk} \, d\ell$$

Boundary condition:

Let $[\xi_i, \xi_j, \xi_k]$ be the face containing ξ_i, ξ_j, ξ_k, and let ν be the outer normal vector. Then

$$p \times \nu = 0 \text{ on } [\xi_i, \xi_j, \xi_k] \iff a_{ij} = a_{ik} = a_{jk} = 0$$

$\Rightarrow$ The boundary condition (3a) can easily be imposed.

(c) Curl conforming finite elements

Choose $V_h := \{ \psi : \Omega \to \mathbb{R}^3 : \psi|_K \text{ constant for all } K \in \mathcal{T}_h \} \subseteq L^2(\Omega)$

$$U_h := \{ \phi : \Omega \to \mathbb{R}^3 : \phi|_K \in \mathcal{R} \ \forall K \in \mathcal{T}_h ,\ \phi \times \nu = 0 \text{ on } \Gamma \}$$

U_h must be curl conforming, i.e. $U_h \subseteq H_0(\text{curl}) = H_0(\text{curl}, \Omega)$.
Is this true?

Let $K_1, K_2 \in \mathcal{T}_h$ with common face $f = \overline{K_1} \cap \overline{K_2}$ and unit normal $\nu : f \to \mathbb{R}^3$. Let $\phi \in U_h$, $\phi_1 := \phi|_{K_1}$, $\phi_2 := \phi|_{K_2}$.

Then $\phi_1 \in H(\text{curl}, K_1)$, $\phi_2 \in H(\text{curl}, K_2)$

Moreover, for every edge of f, the degrees of freedom are the same, i.e.

$$\int_e (\phi_1 - \phi_2) \cdot \tau_e \, dl = 0$$

Lemma 5.35 in Monk's book $\implies$ $\phi_1 \times \nu = \phi_2 \times \nu$ on f

Lemma 5.3 (2) in Monk's book $\implies$ $\phi \in H(\text{curl}, K_1 \cup K_2)$

Hence, $U_h \subseteq H(\text{curl})$.