Karlsruhe Institute of Technology
Department of Mathematics
Prof. Dr. Tobias Jahnke

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Problem class of the lecture

Numerical methods for Maxwell's equations

Tobias Jahnke and Michael Kreim

http://www.math.kit.edu/ianm3/edu/numethmaxwell2014s/en

Problem 1: Boundary conditions of Maxwell's equations

Consider the linear Maxwell equations on $\Omega := (0, L)^3$ as in 2.2 and assume that $\mathcal{E}(t, \cdot)$ and $\mathcal{H}(t, \cdot)$ is a classical solution in $C^1(\Omega \times \Omega)$. Show that the boundary conditions

$$\mathcal{E} \times \nu = 0, \quad (\mu H) \cdot \nu = 0 \quad \text{on } \partial\Omega$$

imply

$$\begin{aligned}
\mathcal{H}_1(t, x) = \mathcal{E}_2(t, x) = \mathcal{E}_3(t, x) = 0 \qquad & \text{if } x_1 \in \{0, L\}, \\
\mathcal{H}_2(t, x) = \mathcal{E}_1(t, x) = \mathcal{E}_3(t, x) = 0 \qquad & \text{if } x_2 \in \{0, L\}, \\
\mathcal{H}_3(t, x) = \mathcal{E}_1(t, x) = \mathcal{E}_2(t, x) = 0 \qquad & \text{if } x_3 \in \{0, L\}.
\end{aligned}$$

Problem 2: Errors of finite difference quotients

Let $u : \mathbb{R} \longrightarrow \mathbb{R}$, $x \in \mathbb{R}$ and $h > 0$. Prove that there are constants C_1, C_2, C_3 such that

$$\left| \frac{u(x+h) - x(x)}{h} - u'(x) \right| \leq C_1 h \tag{a}$$

$$\left| \frac{u(x+h/2) - u(x-h/2)}{h} - u'(x) \right| \leq C_2 h^2 \tag{b}$$

$$\left| \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - u''(x) \right| \leq C_3 h^2 \tag{c}$$

if $u \in C^k(\mathbb{R})$ with sufficiently large $k \in \mathbb{N}$. Specify the minimal value of k in each case.

Problem 3: Discrete divergence

(a) Prove that the discrete divergence

$$\widehat{\text{div}} E^n = \widehat{\partial}_1 E_1^n + \widehat{\partial}_2 E_2^n + \widehat{\partial}_3 E_3^n$$

is well-defined on the grid $[i, j, k]$.

(b) Prove that the discrete divergence

$$\widehat{\mathrm{div}} H^{n+\frac{1}{2}} = \widehat{\partial}_1 H_1^{n+\frac{1}{2}} + \widehat{\partial}_2 H_2^{n+\frac{1}{2}} + \widehat{\partial}_3 H_3^{n+\frac{1}{2}}$$

is well-defined on the grid $[i + \frac{1}{2}, j + \frac{1}{2}, k + \frac{1}{2}]$.

Problem 4: Commutativity of finite differences

Prove Lemma 2.2.

Problem 5: Central finite difference quotient

Show that

$$\widehat{\partial}_1^2 E_j^n[\alpha, \beta, \gamma] = \frac{E_j^n[\alpha+1, \beta, \gamma] - 2E_j^n[\alpha, \beta, \gamma] + E_j^n[\alpha-1, \beta, \gamma]}{h^2}$$

Problem 6: Equivalent formulations of the Störmer-Verlet method

Consider the initial-value problem

$$\begin{aligned} \dot{q} &= v & \qquad q(0) &= q_0 \\ \dot{v} &= f(q) & \qquad v(0) &= v_0 \end{aligned}$$

with some function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$. Show that the one-step formulation (15) of the Störmer-Verlet method is equivalent to the two-step formulation

$$\frac{q^{n+1} - 2q^n + q^{n-1}}{\tau^2} = f(q^n)$$

with initial step $q^1 = q^0 + \tau v^0 + \frac{\tau^2}{2} f(q^0)$.