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Problem class of the lecture

# Numerical methods for Maxwell's equations

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http://www.math.kit.edu/ianm3/edu/numethmaxwell2014s/en

## Problem 7: Stability of the Crank-Nicolson method

Consider the initial-value problem

$$\dot{y}(t) = Ay(t) + f(t) \qquad t \geq 0, \qquad\qquad y(0) = y^0$$

with $A = U\Lambda U^* \in \mathbb{R}^{d\times d}$. Assume that $U \in \mathbb{C}^{d\times d}$ is unitary and that $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^{d\times d}$ is diagonal with $Re(\lambda_k) \leq 0$. Consider the Crank-Nicolson method

$$y^{n+1} = y^n + \frac{\tau}{2}A(y^{n+1} + y^n) + \tau f(t_n + \tau/2), \qquad n = 0, 1, 2, \ldots$$

(a) Assume $f(t) \equiv 0$ and prove that $\|y^n\|_2 \leq \|y^0\|_2$ for all $\mathbb{N}$ and any (arbitrarily large) step-size $\tau > 0$.

(b) What happens if $f(t) = f > 0$ is constant but non-zero?

## Problem 8: Cayley transform

For a skew-adjoint operator $A$ with domain $D(A)$ on a Hilbert space $X$, we define the Cayley transform

$$\Psi(A) = \left(I + \tfrac{1}{2}A\right)\left(I - \tfrac{1}{2}A\right)^{-1}, \qquad D(\Psi) = X.$$

Prove that $\Psi(A) : X \longrightarrow X$ is an isometry. Hint: Show first that $\|\Psi(A)v\| = \|v\|$ for all $v \in D(A)$.

## Problem 9: Recursion for $\Lambda_j$

Let $M$ be the generator of a strongly continuous semigroup $(e^{tM})_{t\geq 0}$ and define

$$\Lambda_0(\tau) = e^{\tau M}, \qquad \Lambda_j(\tau) = \frac{1}{(j-1)!}\int_0^1 (1-s)^{j-1} e^{s\tau M}\, ds.$$

Prove that

$$\tau M \Lambda_{j+1}(s) = \Lambda_j(s) - \frac{1}{j!} I \qquad \text{for all } j \in \mathbb{N}.$$

## Problem 10: Lady Windermere's linear sister

Let $y^n = \Phi_\tau y^{n-1} = \Phi_\tau^n y^0$ be a numerical method for the linear ODE $\dot{y} = My$ with initial value $y(0) = y^0 \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d \times d}$. As usual, $\tau > 0$ is the step-size, $t_n = n\tau$ and $y^n \approx y(t_n)$. Show that

$$y^n - y(t_n) = \sum_{j=0}^{n-1} \Phi_\tau^{n-j-1} \left( \Phi_\tau - e^{\tau M} \right) e^{t_j M} y^0.$$

You do not have to give a formal proof – just write down the first and the last terms of the sum to see what happens.