Stabilität von Runge-Kutta-Verfahren

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Karlsruher Institut für Technologie

Vorlesung *Numerische Methoden für Differentialgleichungen*

Wintersemester 2015/16

Dahlquistsche Testgleichung

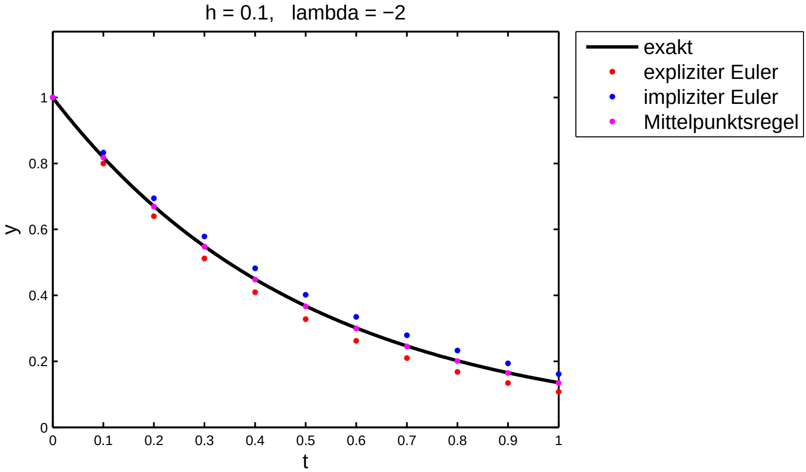
$$\dot{y} = \lambda y, \qquad y(0) = 1$$

Beispiel 1: $\lambda = -2$ nicht steif
Beispiel 2: $\lambda = -100$ steif
Beispiel 3: $\lambda = i$ isometrieerhaltend

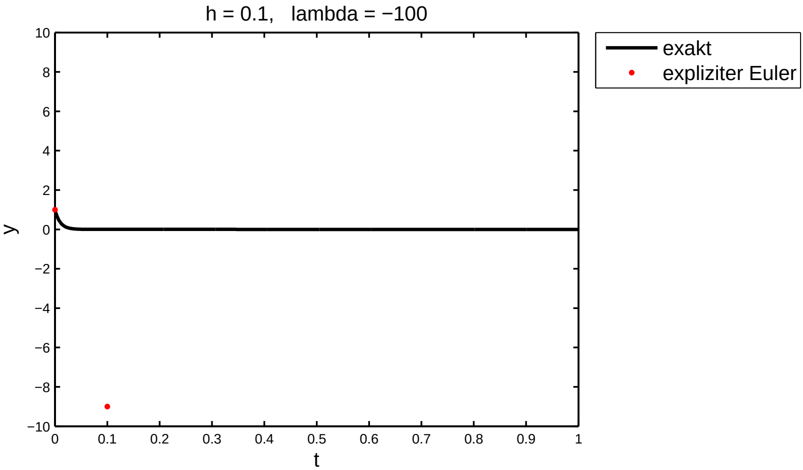
Numerische Verfahren

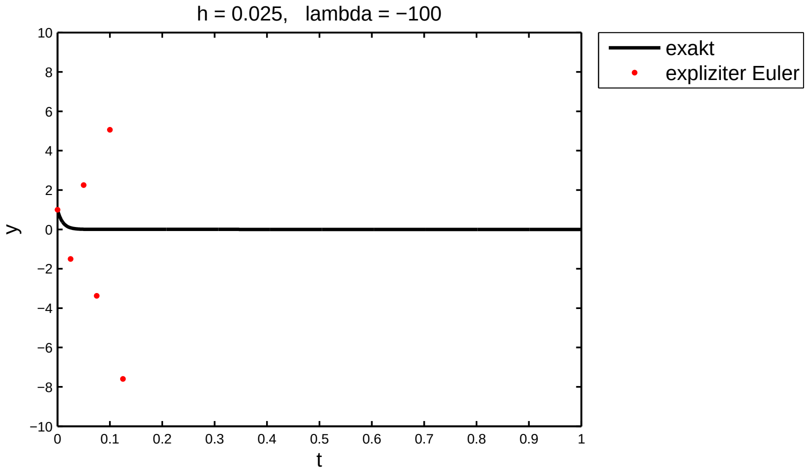
Explizites Euler-Verfahren
Implizites Euler-Verfahren (A-stabil, L-stabil)
Implizite Mittelpunktsregel (A-stabil, isometrieerhaltend)

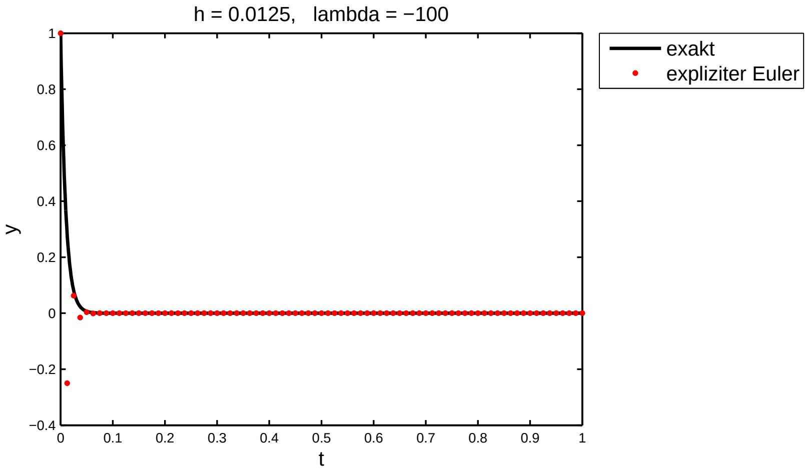
Beispiel 1: $\lambda = -2$

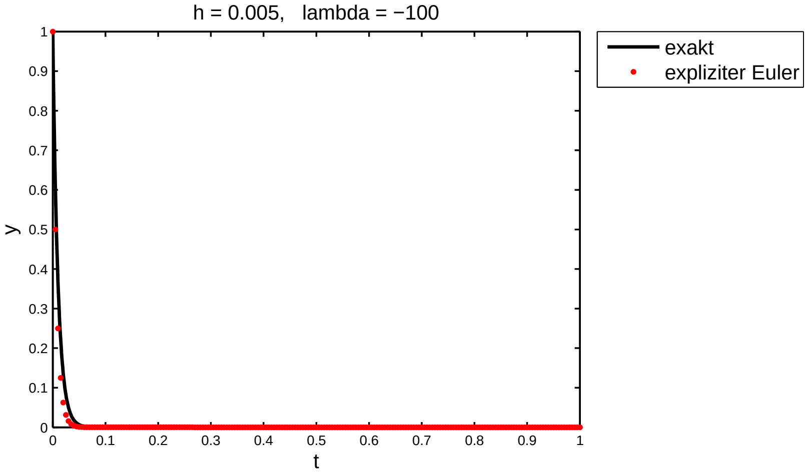

h = 0.1, lambda = −2
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
y
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

Beispiel 2: $\lambda = -100$

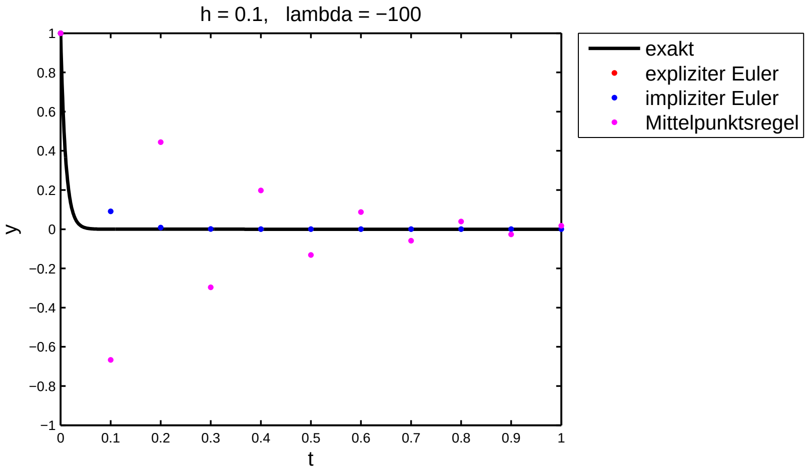

h = 0.1, lambda = −100
10
8
6
4
2
0
−2
−4
−6
−8
−10
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exakt
expliziter Euler

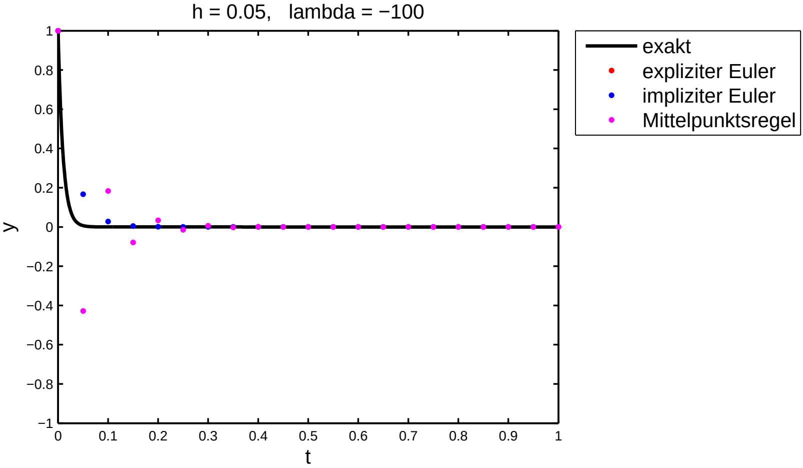

h = 0.025, lambda = −100
10
8
6
4
2
0
−2
−4
−6
−8
−10
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exakt
expliziter Euler

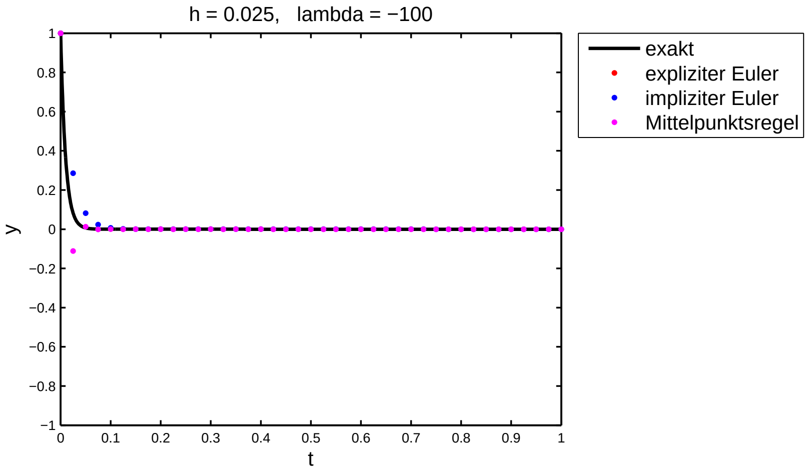

h = 0.0125, lambda = −100
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exakt
expliziter Euler

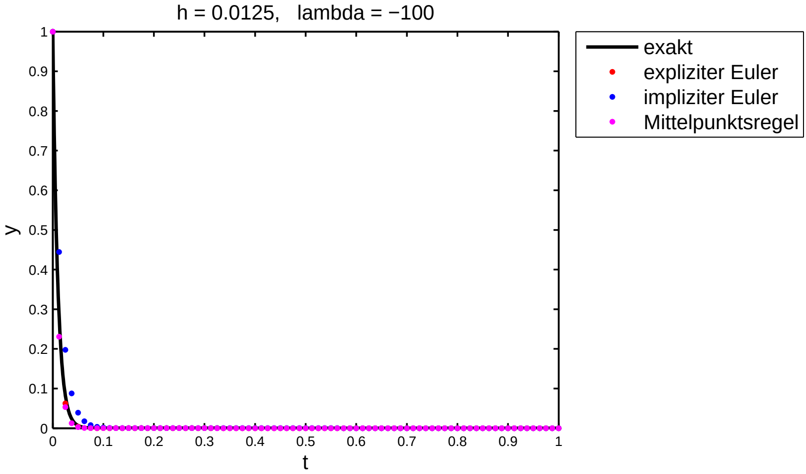

h = 0.005, lambda = −100
exakt
expliziter Euler
y
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Implizites Euler-Verfahren und implizite Mittelpunktsregel


h = 0.1,   lambda = −100
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel


h = 0.05,   lambda = −100
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
y
t
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


h = 0.025, lambda = −100
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel


h = 0.0125,   lambda = −100
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
y
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

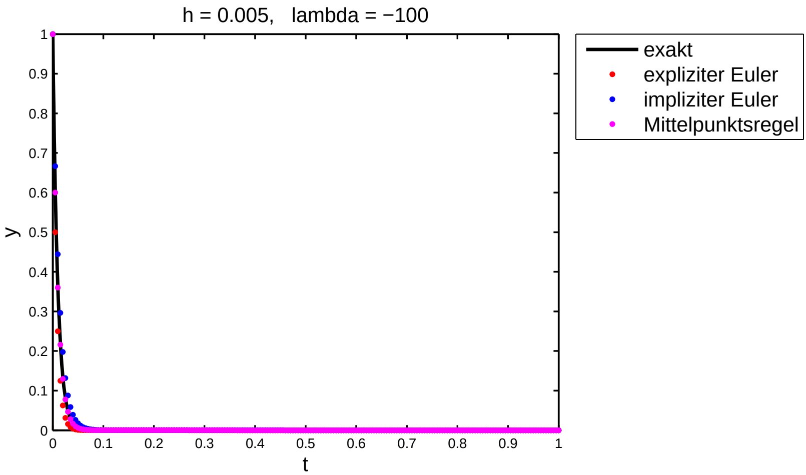
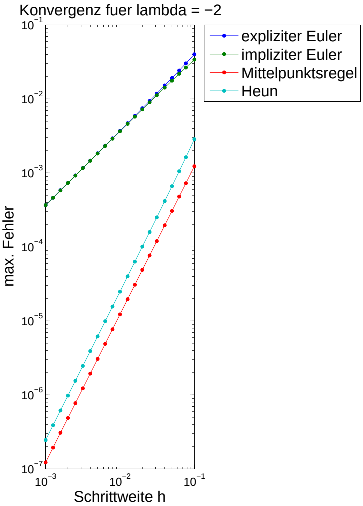
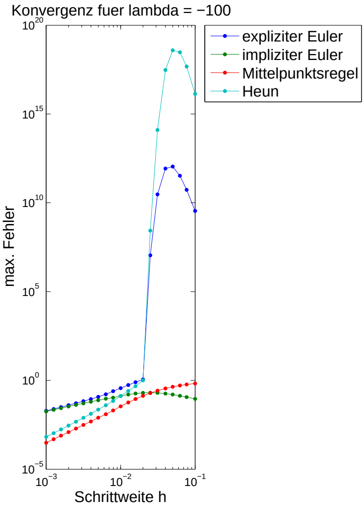
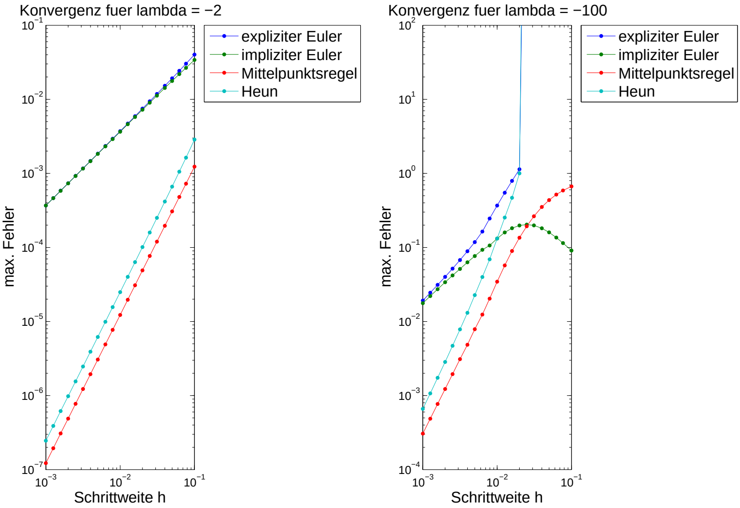

h = 0.005, lambda = −100
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
y
t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

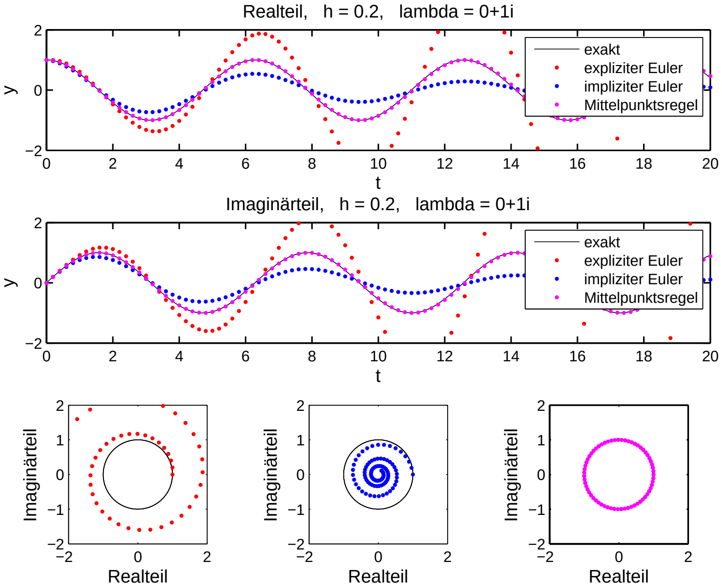
Illustration der Konvergenzordnung

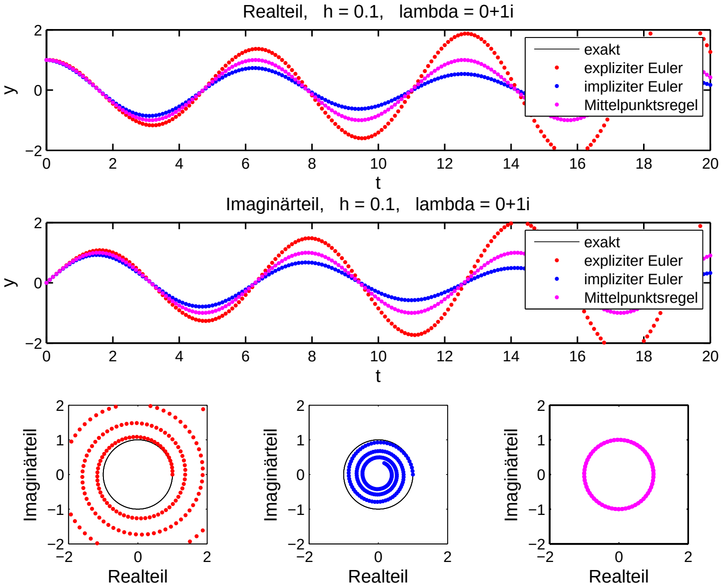

Konvergenz fuer lambda = −2
expliziter Euler
impliziter Euler
Mittelpunktsregel
Heun
max. Fehler
Schrittweite h
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}

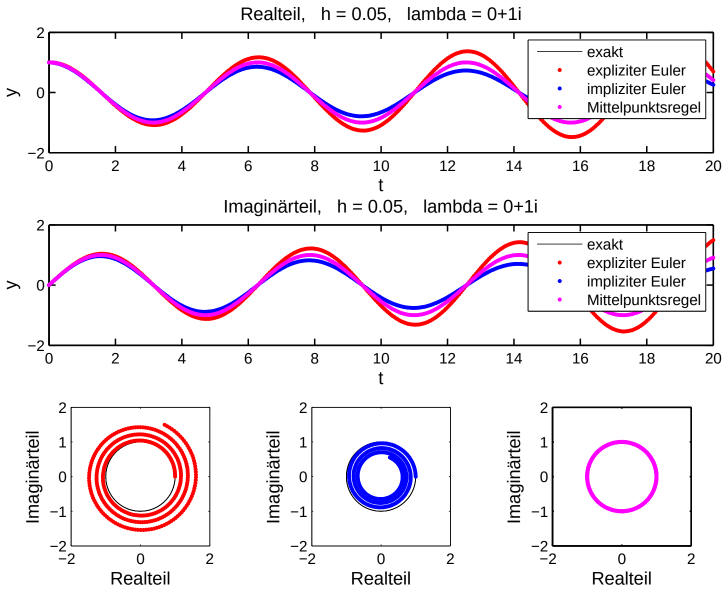

Konvergenz fuer lambda = −100
expliziter Euler
impliziter Euler
Mittelpunktsregel
Heun
max. Fehler
Schrittweite h
10^{20}
10^{15}
10^{10}
10^{5}
10^{0}
10^{-5}
10^{-3}
10^{-2}
10^{-1}


Konvergenz fuer lambda = −2
Konvergenz fuer lambda = −100
expliziter Euler
impliziter Euler
Mittelpunktsregel
Heun
max. Fehler
Schrittweite h
10^−1
10^−2
10^−3
10^−4
10^−5
10^−6
10^−7
10^2
10^1
10^0

Beispiel 3: $\lambda = i$


Realteil, h = 0.2, lambda = 0+1i
y
t
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
Imaginärteil, h = 0.2, lambda = 0+1i
Imaginärteil
Realteil


Realteil, h = 0.1, lambda = 0+1i
y
t
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
Imaginärteil, h = 0.1, lambda = 0+1i
y
t
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
Imaginärteil
Realteil
Imaginärteil
Realteil
Imaginärteil
Realteil


Realteil, h = 0.05, lambda = 0+1i
Imaginärteil, h = 0.05, lambda = 0+1i
exakt
expliziter Euler
impliziter Euler
Mittelpunktsregel
y
t
Imaginärteil
Realteil

Ein Beispiel zur L-Stabilität

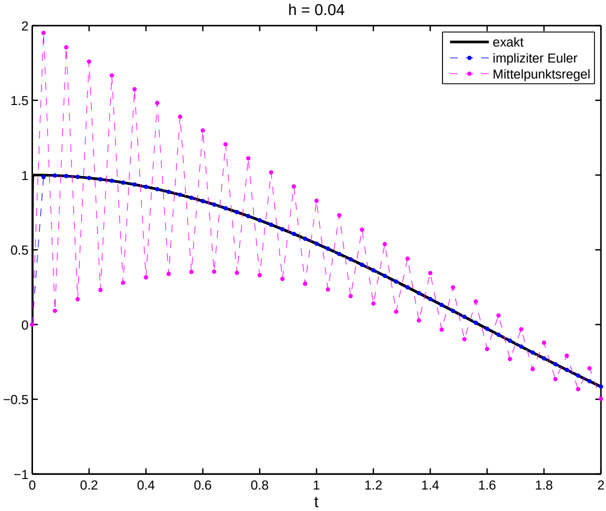
Anfangswertproblem:

$$\dot{y} = -\lambda\big(y(t) - \cos(t)\big), \qquad y(0) = 0, \qquad \lambda \gg 1$$

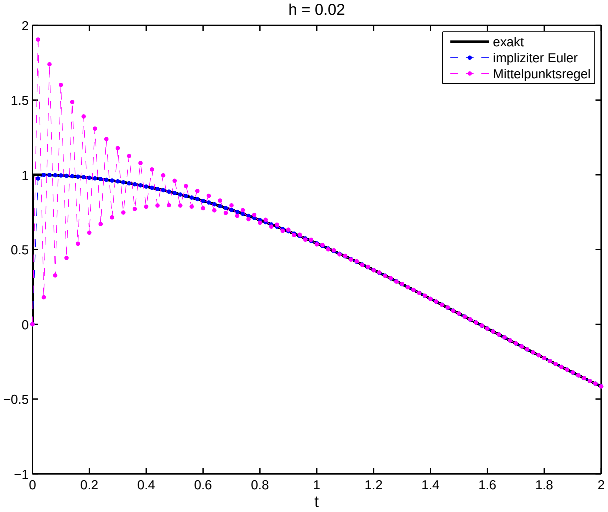
Variation der Konstanten und partielle Integration:

$$\begin{aligned}
y(t) &= e^{-\lambda t} \underbrace{y(0)}_{=0} + \int_0^t e^{-\lambda(t-s)} \lambda \cos(s)\, ds \\
&= \Big[e^{-\lambda(t-s)} \cos(s) \Big]_{s=0}^{t} - \underbrace{\int_0^t e^{-\lambda(t-s)} (-\sin(s))\, ds}_{\mathcal{O}(1/\lambda)} \\
&= \cos(t) - e^{-\lambda t} + \mathcal{O}(1/\lambda)
\end{aligned}$$

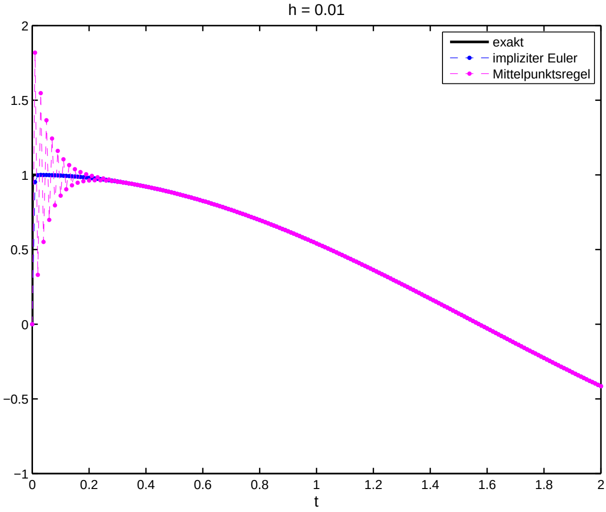
Anfangswertproblem: $\dot{y} = -2000\big(y(t) - \cos(t)\big), \quad y(0) = 0$



Anfangswertproblem: $\dot{y} = -2000\big(y(t) - \cos(t)\big), \quad y(0) = 0$



Anfangswertproblem: $\dot{y} = -2000\big(y(t) - \cos(t)\big), \quad y(0) = 0$



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