Numerical methods in mathematical finance

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Preface

These notes are the basis of my lecture *Numerical methods in mathematical finance* given at Karlsruhe Institute of Technology in the winter term 2014/15, 2016/17, and 2018/19. The purpose of this notes is to help students who have missed parts of the course to fill these gaps, and to provide a service for those students who can concentrate better if they do not have to copy what I write on the blackboard.

It is *not* the purpose of these notes, however, to replace the lecture itself, or to write a text which could compete with the excellent books about the subject. This is why the style of presentation is rather sketchy. As a rule of thumb, one could say that these notes only cover what I *write* during the lecture, but not everything I *say*.

There are still many typos and possibly also other mistakes. Of course, I will try to correct any mistake I find as soon as possible, but please be aware of the fact that you cannot rely on these notes.

Karlsruhe, winter term 2018/19,
Tobias Jahnke
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Chapter 1
Options and arbitrage

References: [BK04, Sey09]

1.1 European options

Financial markets trade investments into stocks of a company, commodities (e.g. oil, gold), bonds, derivatives, etc.

Stocks and commodities are risky assets, because their future value cannot be predicted. Bonds are considered as riskless assets in this lecture. If \( B(t_0) \) is invested at time \( t_0 \) into a bond with a risk-free interest rate \( r > 0 \), then the value of the bond at time \( t \geq t_0 \) is simply

\[
B(t) = e^{r(t-t_0)}B(t_0).
\] (1.1)

Simplifying assumption: continuous payment of interest

Remark. Discrete payment of interest: obtain \( rB(0)\Delta t \) after time \( \Delta t \). Value at \( t = n\Delta t \):

\[
\tilde{B}(t) = (1 + r\Delta t)^nB(0) = (1 + rt/n)^nB(0)
\]

For \( n \rightarrow \infty \) and \( \Delta t \rightarrow 0 \):

\[
\lim_{n \rightarrow \infty} \tilde{B}(t) = \lim_{n \rightarrow \infty} (1 + rt/n)^nB(0) = e^{rt}B(0) = B(t)
\]

(continuous payment of interest)

Spot contract: buy or sell an asset (e.g. a stock, a commodity etc.) with immediate delivery

Financial derivatives: contracts about future payments or deliveries with certain conditions
1. Forwards and futures: agreement between two parties to buy or sell an asset at a certain time in the future for a certain delivery price
2. Swaps: contracts regulating an exchange of cash flows at different future times (e.g. currency swap, interest rate swaps, credit default swaps)
3. Options

Definition 1.1.1 (European option)

- A European call option is a contract which gives the holder (=buyer) of the option the right to buy an underlying risky asset at a future maturity date (expiration time) $T$ at a fixed exercise price (strike) $K$ from the writer (=seller) of the option.
  Typical assets: stocks, parcels of stocks, stock indices, currencies, commodities, ...
  Difference to forwards and futures: At maturity the holder can choose if he wants to buy the asset or not.
- European put option: Similar to call option, but vice versa, i.e. the holder can sell the underlying to the writer.

Example: At time $t = 0$ Mr. J. buys 50 European call options. Each of these options gives him the right to buy one share of the company KIT at maturity $T > 0$ at the exercise price of $K = 120\text{€}$ per share.

- Case 1: At time $t = T$, the market price of KIT is 150€ per share. Mr. J. exercises his options, i.e. he buys 50 KIT shares at the price of $K = 120\text{€}$ per share and sells the shares on the market for 150€ per share. Hence, he wins $50 \cdot (150 - 120) = 1500\text{€}$.
- Case 2: At time $t = T$, the market price of KIT is 100€ per share. Hence, Mr. J. does not exercise his options.

What are options good for?

- Speculation
- Hedging (“insurance” against changing market values)

Since an option gives an advantage to the holder, the option has a certain value. For given $T$ and $K$ the value $V(t, S)$ of the option must depend on the time $t$ and the current price $S$ of the underlying.

For an European option we know that the value at the maturity $T$ is

$$ V(T, S) = \begin{cases} (S - K)^+ := \max\{S - K, 0\} & \text{(European call)} \\ (K - S)^+ := \max\{K - S, 0\} & \text{(European put)} \end{cases} $$

The functions $S \mapsto (S - K)^+$ and $S \mapsto (K - S)^+$ are called the payoff functions of a call or put, respectively.

The goal of this course is to answer the following question:
What is the fair price $V(t, S)$ of an option for $t < T$?

Why is this question important? In order to sell/buy an option, we need to know the fair price.

Why is this question non-trivial? Because the value of the risky asset is random. In particular, the price $S(T)$ at the future expiration time $T$ is not yet known when we buy/sell the option at time $t = 0$.

1.2 More types of options

Variations of the basic principle:

- **European options** can be exercised only at the maturity date.
- **American options** can be exercised at any time before and including the maturity date.
- **Bermuda options** can be exercised at a set of times.

The names “European”, “American”, “Bermuda” etc. have no geographical meaning. American options can be traded in Europe, European options can be traded in the USA, etc.

- **Vanilla options** = standard options, i.e. European, American or Bermuda calls/puts
- **Exotic options** = non-standard options

Examples for exotic options:

- **Path-dependent option**: The payoff function does not only depend on the price $S(T)$ of the underlying at time $T$, but on the entire path $t \mapsto S(t)$ for $t \in [0, T]$.
  - **Asian options**: The payoff function depends on the average price, e.g.
    $$\left( \frac{1}{T} \int_0^T S(t) \, dt - K \right)^+$$
    (payoff of an average price call).
  - **Barrier options**: The payoff depends on the question if the price of the underlying has crossed a certain (upper or lower) barrier.
  - **Lookback options**: The payoff depends on $\max_{t \in [0,T]} S(t)$ or $\min_{t \in [0,T]} S(t)$.

- **Options on several assets**:
  - **Basket options**: The payoff depends on the weighted sum of the prices $S_i$ of several assets, e.g.
    $$\left( \sum_{i=1}^d c_i S_i - K \right)^+, \quad c_i > 0$$
(payoff of a basket call)

- **Rainbow options**: The payoff depends on the relation between the assets, e.g. \( \max\{S_1, \ldots, S_d\} \).
- **Binary options**: The payoff function has only two possible values
- **Compound options**: Options on options

**Remark**: There are even more types of options.

### 1.3 Arbitrage and modelling assumptions

**Example.** Consider

- a stock with price \( S(t) \)
- a European call option with maturity \( T = 1 \), strike \( K = 100 \), and value \( V(t) := V(t, S(t)) \)
- a bond with price \( B(t) \) and interest rate \( r > 0 \)

**Initial data:** \( S(0) = 100, B(0) = 100, V(0) = 10 \).

**Assumption:** At time \( t = 1 \), we either have

- "up": \( B(1) = 110, S(1) = 120 \)
- "down": \( B(1) = 110, S(1) = 80 \)

At \( t = 0 \), Mrs. C. buys 0.4 bonds, one call option and sells 0.5 stock ("short selling").

Value of the portfolio at \( t = 0 \):

\[
0.4 \cdot B(0) + 1 \cdot V(0) - 0.5 \cdot S(0) = 0.4 \cdot 100 + 1 \cdot 10 - 0.5 \cdot 100 = 0
\]

Value of the portfolio at \( t = T = 1 \) is

\[
0.4 \cdot B(1) + 1 \cdot \underbrace{V(1)}_{=(S(1)-K)^+} - 0.5 \cdot S(1)
\]

Two cases:

- "up": \( 0.4 \cdot 110 + 1 \cdot (120 - 100)^+ - 0.5 \cdot 120 = 44 + 20 - 60 = 4 \)
- "down": \( 0.4 \cdot 110 + 1 \cdot (80 - 100)^+ - 0.5 \cdot 80 = 44 + 0 - 40 = 4 \)

In both cases, Mrs. C. wins 4€ without any risk or investment!

**Why is this possible?** Because the price \( V(0) = 10 \) of the option is too low!

**Definition 1.3.1 (Arbitrage)** **Arbitrage** is the existence of a portfolio, which

- requires no initial investment, and
- which cannot cause any loss, but very likely a gain.

**Remark.** A bond will always yield a risk-less gain, but it requires an investment.
Assumptions for modelling an idealized market:

(A1) Arbitrage is impossible (no-arbitrage principle).

(A2) There is a risk-free interest rate $r > 0$ which applies for all credits. Continuous payment of interest according to (1.1).

(A3) No transaction costs, taxes, etc. Trading is possible at any time. Any fraction of an asset can be sold. Liquid market, i.e. selling an asset does not change its value significantly.

(A4) A seller can sell assets he/she does not own yet (“short selling”, cf. Mrs. C. above)

(A5) No dividends on the underlying asset are paid.

1.4 Arbitrage bounds

Consider European options with strike $K > 0$ and maturity $T$ on an underlying with price $S(t)$. Let $V_P(t, S)$ and $V_C(t, S)$ be the values of a put option and call option, respectively.

Lemma 1.4.1 (Put-call parity) Under the assumptions (A1)-(A5) we have

$$S(t) + V_P(t, S(t)) - V_C(t, S(t)) = e^{-r(T-t)}K$$

for all $t \in [0, T]$.

Proof. Buy one stock, buy a put, write (sell) a call. Then, the value of this portfolio is

$$\pi(t) = S(t) + V_P(t, S(t)) - V_C(t, S(t))$$

and at maturity

$$\pi(T) = S(T) + V_P(T, S(T)) - V_C(T, S(T)) = S(T) + (K - S(T))^+ - (S(T) - K)^+ = K.$$ 

Hence, the portfolio is risk-less. No arbitrage: The profit of the portfolio must be the same as the profit for investing $\pi(t)$ into a bond at time $t$:

$$\pi(T) = K = e^{r(T-t)}\pi(t) \implies e^{-r(T-t)}K = \pi(t) = S(t) + V_P(t, S(t)) - V_C(t, S(t)).$$

Lemma 1.4.2 (Bounds for European calls and puts) Under the assumptions (A1)-(A5), the following inequalities hold for all $t \in [0, T]$ and all $S = S(t) \geq 0$:

$$S - e^{-r(T-t)}K \leq V_C(t, S) \leq S$$

$$e^{-r(T-t)}K - S \leq V_P(t, S) \leq e^{-r(T-t)}K$$

Proof.

• It is obvious that $V_C(t, S) \geq 0$ and $V_P(t, S) \geq 0$ for all $t \in [0, T]$ and $S \geq 0$. 

Assume that \( V_C(t, S(t)) > S(t) \) for some \( S(t) \geq 0 \).
Write (sell) a call, buy the stock and put the difference \( \delta := V_C(t, S(t)) - S(t) > 0 \) in your pocket.
At \( t = T \), there are two scenarios:

If \( S(T) > K \): Must sell stock at the price \( K \) to the owner of the call.
   Gain: \( K + \delta > 0 \)
If \( S(T) \leq K \): Gain \( S(T) + \delta > 0 \)
\( \implies \) Arbitrage! Contradiction!

Put-call parity:
\[
S - e^{-r(T-t)}K = V_C(t, S) - V_P(t, S) \geq 0
\]

This proves (1.2). The proof of (1.3) is left as an exercise.

**Remark.** Similar inequalities can be shown for American options (exercise).

### 1.5 A simple discrete model

Consider
- a stock with price \( S(t) \)
- a European option with maturity \( T \), strike \( K \), and value \( V(t, S(t)) \)
- a bond with price \( B(t) = e^{rt}B(0) \)

Suppose that the initial data \( S(0) = S_0 \) and \( B(0) = 1 \) are known, and that (A1)-(A5) hold. Goal: Find \( V(0, S_0) \).

Simplifying assumption: At time \( t = T \), there are only two scenarios

\begin{align*}
\text{“up”: } & S(T) = u \cdot S_0 & \text{with probability } p \\
\text{“down”: } & S(T) = d \cdot S_0 & \text{with probability } 1 - p
\end{align*}

Assumption: \( 0 < d \leq e^{rT} \leq u \) and \( p \in (0, 1) \)

In both cases, we have \( B(T) = e^{rT}B(0) = e^{rT} \).

**Replication strategy:** Construct portfolio with \( c_1 \) bonds and \( c_2 \) stocks such that
\[
c_1B(t) + c_2S(t) \overset{!}{=} V(t, S(t))
\]
for \( t \in \{0, T\} \). For \( t = T \), this means
\[
\text{case “up”: } c_1e^{rT} + c_2uS_0 = V(T, uS_0) =: V_u \\
\text{case “down”: } c_1e^{rT} + c_2dS_0 = V(T, dS_0) =: V_d
\]
$V_u$ and $V_d$ are known if $u$ and $d$ are known. The unique solution is (check!)

$$c_1 = \frac{uV_d - dV_u}{(u - d)e^{rT}} \quad c_2 = \frac{V_u - V_d}{(u - d)S_0}.$$ 

Hence, the fair price of the option is

$$V(0, S_0) = c_1 B(0) + c_2 S_0 = \frac{uV_d - dV_u}{(u - d)e^{rT}} + \frac{V_u - V_d}{(u - d)}$$

which yields (check!)

$$V(0, S_0) = e^{-rT}(qV_u + (1 - q)V_d) \quad \text{with} \quad q := \frac{e^{rT} - d}{u - d}.$$ (1.4)

**Remark:** The value of the option does not depend on $p$.

Since $0 < d \leq e^{rT} \leq u$ by assumption, $q \in [0, 1]$ can be seen as a probability. Now, define a new probability distribution $\mathbb{P}_q$ by

$$\mathbb{P}_q\left(S(T) = uS_0\right) = q, \quad \mathbb{P}_q\left(S(T) = dS_0\right) = 1 - q$$

($q$ instead of $p$). Then, we have

$$\mathbb{P}_q\left(V(T, S(T)) = V_u\right) = q, \quad \mathbb{P}_q\left(V(T, S(T)) = V_d\right) = 1 - q$$

and hence

$$qV_u + (1 - q)V_d = \mathbb{E}_q\left(V(T, S(T))\right)$$

can be regarded as the expectation of the payoff $V(T, S(T))$ with respect to $\mathbb{P}_q$. In (1.4), this expectation is multiplied by an discounting factor $e^{-rT}$.

**Interpretation:** In order to have an amount of $B(t)$ at time $t$, we have to invest $B(0) = e^{-rt}B(t)$ into a bond at time $t = 0$.

The probability $q$ has the property that

$$\mathbb{E}_q(S(T)) = quS_0 + (1 - q)dS_0 = \frac{e^{rT} - d}{u - d}uS_0 + \frac{u - e^{rT}}{u - d}dS_0 = e^{rT}S_0.$$ 

Hence, the expected (with respect to $\mathbb{P}_q$) value of $S(T)$ is exactly the amount we obtain when we invest $S_0$ into a bond. Therefore, $\mathbb{P}_q$ is called the **risk-neutral probability**.

**Moral of the story so far:**

**Under the risk-neutral probability, the price of a European option is the discounted expectation of the payoff.**
Chapter 2

Stochastic differential equations

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space\(^1\): \(\Omega \neq \emptyset\) is a set, \(\mathcal{F}\) is a \(\sigma\)-algebra (or \(\sigma\)-field) on \(\Omega\), and \(\mathbb{P}: \mathcal{F} \rightarrow [0, 1]\) is a probability measure.

A probability space is complete if \(\mathcal{F}\) contains all subsets \(G\) of \(\Omega\) with \(\mathbb{P}\)-outer measure zero, i.e. with

\[
\mathbb{P}^*(G) := \inf \{\mathbb{P}(F) : F \in \mathcal{F} \text{ and } G \subset F\} = 0.
\]

Any probability space can be completed. Hence, we can assume that every probability space in this lecture is complete.

2.1 Stochastic processes and filtrations

Definition 2.1.1 (Stochastic process) Let \(\mathcal{T}\) be an ordered set (e.g. \(\mathcal{T} = [0, \infty)\), \(\mathcal{T} = \mathbb{N}\)). A stochastic process is a family \(X = \{X_t : t \in \mathcal{T}\}\) of random variables \(X_t : \Omega \rightarrow \mathbb{R}^d\).

Equivalent notations: \(X(t, \omega), X(t), X_t(\omega), X_t, \ldots\)

Below, we will often simply write \(X_t\) instead of \(\{X_t : t \in \mathcal{T}\}\).

For a fixed \(\omega \in \Omega\), the function \(t \mapsto X_t(\omega)\) is called a realization (or path or trajectory) of \(X\).

The path of a stochastic process is associated to some \(\omega \in \Omega\). As time evolves, more information about \(\omega\) becomes available.

Example (cf. chapter 2 in [Shr04]). Toss a coin three times. Possible results are:

<table>
<thead>
<tr>
<th>(\omega_1)</th>
<th>(\omega_2)</th>
<th>(\omega_3)</th>
<th>(\omega_4)</th>
<th>(\omega_5)</th>
<th>(\omega_6)</th>
<th>(\omega_7)</th>
<th>(\omega_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HHH</td>
<td>HHT</td>
<td>HTH</td>
<td>HTT</td>
<td>THH</td>
<td>THT</td>
<td>TTH</td>
<td>TTT</td>
</tr>
</tbody>
</table>

\(H = \text{heads, } T = \text{tails}\).

\(^1\)See “2.2.2 What is \((\Omega, \mathcal{F}, \mathbb{P})\) anyway?” in the book [CT04] for a nice discussion of this concept.
Before the first toss, we only know that \( \omega \in \Omega = \{\omega_1, \ldots, \omega_8\} \).

After the first toss, we know if the final result will belong to
\[
\{HHH, HHT, HTH, HTT\} \text{ or to } \{THH, THT, TTH, TTT\}.
\]
These sets are “resolved by the information”. Hence, we know in which of the sets
\[
\{w_1, w_2, w_3, w_4\}, \{w_5, w_6, w_7, w_8\}
\]
\( \omega \) is.

After the second toss, the sets
\[
\{HHH, HHT\}, \{HTH, HTT\}, \{THH, THT\}, \{TTH, TTT\}
\]
are resolved, and we know in which of the sets
\[
\{w_1, w_2\}, \{\omega_3, w_4\}, \{w_5, w_6\}, \{w_7, w_8\}
\]
\( \omega \) is.

This motivates the following definition.

**Definition 2.1.2 (Filtration)**

- A **filtration** is a family \( \{\mathcal{F}_t : t \geq 0\} \) of sub-\( \sigma \)-algebras of \( \mathcal{F} \) such that \( \mathcal{F}_s \subseteq \mathcal{F}_t \) for all \( t \geq s \geq 0 \).

  A filtration models the fact that more and more information about a process is known as time evolves.

- If \( \{X_t : t \geq 0\} \) is a family of random variables and \( X_t \) is \( \mathcal{F}_t \)-measurable, then \( \{X_t : t \geq 0\} \) is adapted to (or nonanticipating with respect to) \( \{\mathcal{F}_t : t \geq 0\} \).

  Interpretation: At time \( t \) we know for each set \( S \in \mathcal{F}_t \) if \( \omega \in S \) or not. The value of \( X_t \) is revealed at time \( t \).

- For every \( s \in [0, t] \) let \( \sigma\{X_s\} \) be the \( \sigma \)-algebra generated by \( X_s \), i.e. the smallest \( \sigma \)-algebra on \( \Omega \) containing the sets
\[
X_s^{-1}(B) \text{ for all } B \in \mathcal{B}
\]
where \( \mathcal{B} \) denotes the Borel \( \sigma \)-algebra. By definition \( \sigma\{X_s\} \) is the smallest \( \sigma \)-algebra where \( X_s \) is measurable.

**2.2 The Wiener process**

Robert Brown 1827, Louis Bachelier 1900, Albert Einstein 1905, Norbert Wiener 1923
Definition 2.2.1 (Normal distribution) A random variable $X : \Omega \rightarrow \mathbb{R}^d$ with $d \in \mathbb{N}$ is normal if it has a multivariate normal (Gaussian) distribution with mean $\mu \in \mathbb{R}^d$ and a symmetric, positive definite covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$, i.e.

$$
\mathbb{P}(X \in B) = \int_B \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) dx
$$

for all Borel sets $B \subset \mathbb{R}^d$. Notation: $X \sim \mathcal{N}(\mu, \Sigma)$

Remarks:
1. If $X \sim \mathcal{N}(\mu, \Sigma)$, then $\mathbb{E}(X) = \mu$ and $\Sigma = (\sigma_{ij})$ with $\sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$.
2. Standard normal distribution $\Leftrightarrow \mu = 0, \Sigma = I$ (identity matrix).
3. If $X \sim \mathcal{N}(\mu, \Sigma)$ and $Y = v + TX$ for some $v \in \mathbb{R}^d$ and a regular matrix $T \in \mathbb{R}^{d \times d}$, then

$$
Y \sim \mathcal{N}(v + T\mu, T\Sigma T^T).
$$

(2.1)

4. Warning: In one dimension, the covariance matrix is simply a number, namely the variance. Unfortunately, the variance is usually denoted by $\sigma^2$ instead of $\sigma$ in the literature, which is somewhat confusing.

Definition 2.2.2 (Wiener process, Brownian motion)

(a) A stochastic process $\{W_t : t \in [0,T]\}$ is called a standard Brownian motion or standard Wiener process if it has the following properties:

1. $W_0 = 0$ (with probability one)
2. Independent increments: For all $0 \leq t_1 < t_2 < \ldots < t_n < T$ the random variables

$$
W_{t_2} - W_{t_1}, \quad W_{t_3} - W_{t_2}, \quad \ldots, \quad W_{t_n} - W_{t_{n-1}}
$$

are independent.
3. $W_t - W_s \sim \mathcal{N}(0, t - s)$ for any $0 \leq s < t < T$.
4. There is a $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that $t \mapsto W_t(\omega)$ is continuous for all $\omega \in \tilde{\Omega}$.

(b) If $W_t^{(1)}, \ldots, W_t^{(d)}$ are independent one-dimensional Wiener processes, then $W_t = \left(W_t^{(1)}, \ldots, W_t^{(d)}\right)$ is called a d-dimensional Wiener process, and

$$
W_t - W_s \sim \mathcal{N}(0, (t - s)I).
$$

Existence of Brownian motion was first proved in a mathematically rigorous way by Norbert Wiener in 1923.

The Wiener process will serve as the “source of randomness” in our model of the financial market.
Notation: \( W_t = W_t(\omega) = W(t, \omega) = W(t) \)

Numerical simulation of a Wiener process \((d=1)\). Choose step-size \( \tau > 0 \), put \( t_n = n\tau \) and \( \tilde{W}_0 = 0 \).

for \( n = 0, 1, 2, 3, \ldots \)
  Generate random number \( Z_n \sim \mathcal{N}(0, 1) \)
  \( \tilde{W}_{n+1} = \tilde{W}_n + \sqrt{\tau}Z_n \)
end for

For \( \tau \rightarrow 0 \) the interpolation of \( \tilde{W}_0, \tilde{W}_1, \tilde{W}_2, \ldots \) approximates a path of the Wiener process \( (\tilde{W}_N \approx W_{n\tau}) \).

How smooth is a path of a Wiener process? Consider only \( d = 1 \).

Hölder continuity and non-differentiability

Definition 2.2.3 (Hölder continuity) A function \( f : (a, b) \rightarrow \mathbb{R} \) is Hölder continuous of order \( \alpha \) for some \( \alpha \in [0, 1] \) if there is a constant \( C \) such that

\[ |f(t) - f(s)| \leq C|t - s|^{\alpha} \quad \text{for all } s, t \in (a, b). \]

If \( \alpha = 0 \), then \( f \) is bounded.
If \( \alpha > 0 \), then \( f \) is uniformly continuous.
If \( \alpha = 1 \), then \( f \) is Lipschitz continuous.

A path of the Wiener process on a bounded interval is
  - Hölder continuous of order \( \alpha \in [0, \frac{1}{2}) \) with probability one, but
  - not Hölder continuous of order \( \alpha \geq \frac{1}{2} \) with probability one.

A path of the Wiener process is nowhere differentiable with probability one.

Proofs: [Ste01], chapter 5

Unbounded total variation

Definition 2.2.4 (Total variation) Let \([a, b] \) be an interval and let

\[ P = \{t_0, t_1, \ldots, t_{N(P)}\}, \quad a = t_0 < t_1 < \ldots < t_{N(P)} = b, \quad N(P) \in \mathbb{N} \]

be a partition of this interval. Let \( \mathcal{P} \) be the set of all such partitions. The total variation of a function \( f : [a, b] \rightarrow \mathbb{R} \) is

\[ TV_{a,b}(f) = \sup_{P \in \mathcal{P}} \sum_{n=1}^{N(P)} |f(t_n) - f(t_{n-1})|. \quad (2.2) \]
If $f$ is differentiable and $f'$ is integrable, then it can be shown that
\[
TV_{a,b}(f) = \int_{a}^{b} |f'(t)| \, dt
\]
Conversely: If a function $f$ has bounded total variation, then its derivative exists for almost all $t \in [a, b]$.
Consequence: A path of the Wiener process has unbounded total variation with probability one.

**Filtration of the Wiener process**

The natural filtration of the Wiener process on $[0,T]$ is given by
\[
\{\mathcal{F}_t : t \in [0,T]\}, \quad \mathcal{F}_t = \sigma\{W_s, s \in [0,t]\}
\]
(cf. Definition 2.1.2). $\mathcal{F}_t$ contains all information (but not more) which can be obtained by observing $W_s$ on the interval $[0,t]$. For technical reasons, however, it is more advantageous to use an augmented filtration called the standard Brownian filtration. See pp. 50-51 in [Ste01] for details.

**2.3 Construction of the Itô integral (step 1 and 2)**

**References:** [KP99, Øks03, Shr04, Ste01]

The model considered in 1.5 is clearly too simple: only two discrete times, only two possible prices of $S(T)$.

**Goal:** Construct a more realistic model for the dynamics of $S(t)$.

Naive Ansatz:
\[
\frac{dX}{dt} = f(t, X) + g(t, X)Z(t), \quad Z(t) = ?
\]
Apply explicit Euler method: Choose $t \geq 0$ and $N \in \mathbb{N}$, let $\tau = t/N$, $t_n = n\tau$ and define approximations $X_n \approx X(t_n)$ by
\[
X_{n+1} = X_n + \tau f(t_n, X_n) + \tau g(t_n, X_n)Z(t_n) \quad (n = 0, 1, 2, \ldots)
\]
with initial data $X_0 = X(0)$. In the special case $f(t, X) = 0$, $g(t, X) = 1$ and $X(0) = 0$, we want that $X_n = W(t_n)$ is the Wiener process, i.e. we postulate that
\[
W(t_{n+1}) = W(t_n) + \tau Z(t_n).
\]
This yields
\[ X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n) \left( W(t_{n+1}) - W(t_n) \right) \]
and after \( N \) steps
\[ X_N = X_0 + \tau \sum_{n=0}^{N-1} f(t_n, X_n) + \sum_{n=0}^{N-1} g(t_n, X_n) \left( W(t_{n+1}) - W(t_n) \right). \tag{2.3} \]

Keep \( t \) fixed, let \( N \to \infty \), \( \tau = t/N \to 0 \). Then, (2.3) should somehow converge to
\[ X(t) = X(0) + \int_0^t f(s, X(s)) \, ds + \int_0^t g(s, X(s)) \, dW(s) . \tag{2.4} \]

**Problem:** We cannot define (★) as a pathwise Riemann-Stieltjes integral (cf. appendix B). When \( N \to \infty \), the sum
\[ \sum_{n=0}^{N-1} g(t_n, X_n(\omega)) \left( W(t_{n+1}, \omega) - W(t_n, \omega) \right) \]
diverges with probability one, because a path of the Wiener process has unbounded total variation with probability one.

**New goal:** Define the integral
\[ I_t[u](\omega) = \int_0^t u(s, \omega) \, dW_s(\omega) \]
in a “reasonable” way for the following class of functions.

**Definition 2.3.1** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \( \{\mathcal{F}_t : t \in [0, T]\} \) be the standard Brownian filtration. Then, we define \( \mathcal{H}^2[0, T] \) to be the class of functions
\[ u = u(t, \omega), \quad u : [0, T] \times \Omega \to \mathbb{R} \]
with the following properties:

1. \((t, \omega) \mapsto u(t, \omega)\) is \((\mathcal{B} \times \mathcal{F})\)-measurable.
2. \( u \) is adapted to \( \{\mathcal{F}_t : t \in [0, T]\} \), i.e. \( u(t, \cdot) \) is \( \mathcal{F}_t \)-measurable.
3. \( \mathbb{E} \left( \int_0^T u^2(t, \omega) \, dt \right) < \infty \)
Step 1: Itô integral for elementary functions

Definition 2.3.2 (Elementary functions) A function \( \phi \in \mathcal{H}^2[0,T] \) is called elementary if it is a stochastic step function of the form

\[
\phi(t, \omega) = a_0(\omega)1_{[0,0]}(t) + \sum_{n=0}^{N-1} a_n(\omega)1_{(t_n,t_{n+1}]}(t)
\]

with a partition \( 0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T \). The random variables \( a_n \) must be \( \mathcal{F}_{t_n} \)-measurable with \( \mathbb{E}(a_n^2) < \infty \). Here and below,

\[
1_{[c,d]}(t) = \begin{cases} 
1 & \text{if } t \in [c,d] \\
0 & \text{else}
\end{cases}
\]

is the indicator function of an interval \([c,d]\).

For \( 0 \leq c < d \leq T \), the only reasonable way to define the Itô integral of an indicator function \( 1_{[c,d]} \) is

\[
\mathcal{I}_T[1_{[c,d]}](\omega) = \int_0^T 1_{[c,d]}(s) \, dW(s, \omega) = \int_c^d dW(s, \omega) = W(d, \omega) - W(c, \omega).
\]

Hence, by linearity, we define the Itô integral of an elementary function by

\[
\mathcal{I}_T[\phi](\omega) = \sum_{n=0}^{N-1} a_n(\omega)\left( W(t_{n+1}, \omega) - W(t_n, \omega) \right).
\]

Lemma 2.3.3 (Itô isometry for elementary functions) For all elementary functions we have

\[
\mathbb{E}\left( \mathcal{I}_T[\phi]^2 \right) = \mathbb{E}\left( \int_0^T \phi^2(t, \omega) \, dt \right)
\]

or equivalently

\[
\|\mathcal{I}_T[\phi]\|_{L^2(\mathbb{P})} = \|\phi\|_{L^2(dt \times d\mathbb{P})}
\]

with

\[
\|\phi\|_{L^2(dt \times d\mathbb{P})} = \left( \int_{\Omega} \left( \int_0^T \phi^2(t, \omega) \, dt \right) d\mathbb{P} \right)^\frac{1}{2} = \left( \mathbb{E}\left( \int_0^T \phi^2(t, \omega) \, dt \right) \right)^\frac{1}{2}.
\]
Proof. Since
\[ \phi^2(t, \omega) = a_0^2(\omega)1_{[0,0]}(t) + \sum_{n=0}^{N-1} a_n^2(\omega)1_{(t_n, t_{n+1}]}(t) \]
we obtain
\[
\mathbb{E} \left( \int_0^T \phi^2(t, \omega) \, dt \right) = \sum_{n=0}^{N-1} \mathbb{E} \left( a_n^2 \right) (t_{n+1} - t_n)
\]
for the right-hand side. If we let \( \Delta W_n = W(t_{n+1}) - W(t_n) \), then
\[
\mathbb{E} \left( I_T \phi \right)^2 = \mathbb{E} \left( \sum_{n=0}^{N-1} a_n \Delta W_n \right)^2 = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mathbb{E}(a_n a_m \Delta W_m \Delta W_n)
\]
(2.7)

• If \( n > m \), then \( a_n a_m \Delta W_m \) is \( \mathcal{F}_{t_n} \)-measurable. \( \Delta W_n \) is independent of \( \mathcal{F}_{t_n} \) because the Wiener process has independent increments. Hence, \( a_n a_m \Delta W_m \) and \( \Delta W_n \) are independent. It follows that
\[
\mathbb{E}(a_n a_m \Delta W_n \Delta W_m) = \mathbb{E}(a_n a_m \Delta W_m) \mathbb{E}(\Delta W_n) = 0 \quad \text{for } n > m
\]
because \( \Delta W_n \sim \mathcal{N}(0, t_{n+1} - t_n) \) by definition.
• By the same argument \( a_n^2 \) and \( \Delta W_n^2 \) are independent. Hence, we obtain
\[
\mathbb{E}(a_n^2 \Delta W_n^2) = \mathbb{E}(a_n^2) (t_{n+1} - t_n)
\]
because \( \mathbb{E}(\Delta W_n^2) = \mathbb{V}(\Delta W_n) = t_{n+1} - t_n \).

Hence, (2.7) simplifies to
\[
\mathbb{E} \left( I_T \phi \right)^2 = \sum_{n=0}^{N-1} \mathbb{E} \left( a_n^2 \right) (t_{n+1} - t_n).
\]
(2.8)

Comparing (2.6) and (2.8) yields the assertion.

Step 2: Itô integral on \( \mathcal{H}^2[0, T] \)

Lemma 2.3.4 For any \( u \in \mathcal{H}^2[0, T] \) there is a sequence \( (\phi_k)_{k \in \mathbb{N}} \) of elementary functions \( \phi_k \in \mathcal{H}^2[0, T] \) such that
\[
\lim_{k \to \infty} \| u - \phi_k \|_{L^2(dt \times dP)} = 0
\]
Proof: Section 6.6 in [Ste01].

Let \( u \in \mathcal{H}^2[0,T] \) and let \((\phi_k)_{k \in \mathbb{N}}\) be elementary functions such that
\[
u = \lim_{k \to \infty} \phi_k \quad \text{in} \ L^2(dt \times d\mathbb{P})
\]
as in Lemma 2.3.4. The linearity of \( \mathcal{I}_T[\cdot] \) and Lemma 2.3.3 yield
\[
\| \mathcal{I}_T[\phi_j] - \mathcal{I}_T[\phi_k] \|_{L^2(d\mathbb{P})} = \| \mathcal{I}_T[\phi_j - \phi_k] \|_{L^2(d\mathbb{P})} = \| \phi_j - \phi_k \|_{L^2(dt \times d\mathbb{P})} \to 0
\]
for \( j, k \to \infty \). Hence, \((\mathcal{I}_T[\phi_k])_k\) is a Cauchy sequence in the Hilbert space \( L^2(d\mathbb{P}) \). Thus, \((\mathcal{I}_T[\phi_k])_k\) converges in \( L^2(d\mathbb{P}) \), and we can define
\[
\mathcal{I}_T[u] = \lim_{k \to \infty} \mathcal{I}_T[\phi_k].
\]

The choice of the sequence does not matter: If \((\psi_k)_{k \in \mathbb{N}}\) is another sequence of elementary functions with
\[
u = \lim_{k \to \infty} \psi_k \quad \text{in} \ L^2(dt \times d\mathbb{P});
\]
cf. Lemma 2.3.4. Then
\[
\lim_{k \to \infty} \| \phi_k - \psi_k \|_{L^2(dt \times d\mathbb{P})} = 0
\]
and hence
\[
\| \mathcal{I}_T[\phi_k] - \mathcal{I}_T[\psi_k] \|_{L^2(d\mathbb{P})} \to 0.
\]

Theorem 2.3.5 (Itô isometry) For all \( u \in \mathcal{H}^2[0,T] \) we have
\[
\| \mathcal{I}_T[u] \|_{L^2(d\mathbb{P})} = \| u \|_{L^2(dt \times d\mathbb{P})}.
\]

Proof: Let \((\phi_k)_{k \in \mathbb{N}}\) again be elementary functions such that \( u = \lim_{k \to \infty} \phi_k \) in \( L^2(dt \times d\mathbb{P}) \); cf. Lemma 2.3.4. Then
\[
\lim_{k \to \infty} \| \phi_k \|_{L^2(dt \times d\mathbb{P})} = \| u \|_{L^2(dt \times d\mathbb{P})},
\]
because the reverse triangle inequality yields
\[
\left| \| \phi_k \|_{L^2(dt \times d\mathbb{P})} - \| u \|_{L^2(dt \times d\mathbb{P})} \right| \leq \| \phi_k - u \|_{L^2(dt \times d\mathbb{P})} \to 0.
\]

By the same argument, we obtain
\[
\lim_{k \to \infty} \| \mathcal{I}_T[\phi_k] \|_{L^2(d\mathbb{P})} = \| \mathcal{I}_T[u] \|_{L^2(d\mathbb{P})}
\]
Now the assertion follows from Lemma 2.3.3 by taking the limit. \( \blacksquare \)
2.4 Martingales

Definition 2.4.1 (conditional expectation) Let $X$ be an integrable random variable, and let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then, $Y$ is a conditional expectation of $X$ with respect to $\mathcal{G}$ if $Y$ is $\mathcal{G}$-measurable and if

$$E(X1_A) = E(Y1_A) \quad \text{for all } A \in \mathcal{G},$$

$$\Leftrightarrow \int_A X(\omega) \, dP(\omega) = \int_A Y(\omega) \, dP(\omega) \quad \text{for all } A \in \mathcal{G}.$$ 

In this case, we write $Y = E(X \mid \mathcal{G})$.

“This definition is not easy to love. Fortunately, love is not required.”

J.M. Steele in [Ste01], p. 45.

Interpretation. $E(X \mid \mathcal{G})$ is a random variable on $(\Omega, \mathcal{G}, P)$ and hence on $(\Omega, \mathcal{F}, P)$, too. Roughly speaking, $E(X \mid \mathcal{G})$ is the best approximation of $X$ detectable by the events in $\mathcal{G}$. The more $\mathcal{G}$ is refined, the better $E(X \mid \mathcal{G})$ approximates $X$.

Examples.

1. If $\mathcal{G} = \{\Omega, \emptyset\}$, then $E(X \mid \mathcal{G}) = E(X) = \int_\Omega X(\omega) \, dP(\omega)$.
2. If $\mathcal{G} = \mathcal{F}$, then $E(X \mid \mathcal{G}) = X$.
3. If $F \in \mathcal{F}$ with $P(F) > 0$ and
   $$\mathcal{G} = \{\emptyset, F, \Omega \setminus F, \Omega\},$$
   then it can be shown that
   $$E(X \mid \mathcal{G})(\omega) = \begin{cases} \frac{1}{P(F)} \int_F X \, dP & \text{if } \omega \in F \\ \frac{1}{P(\Omega \setminus F)} \int_{\Omega \setminus F} X \, dP & \text{if } \omega \in \Omega \setminus F. \end{cases}$$
4. If $X$ is independent of $\mathcal{G}$, then $E(X \mid \mathcal{G}) = E(X)$.

Proof: Exercise.

Lemma 2.4.2 (Properties of the conditional expectation) For all integrable random variables $X$ and $Y$ and all sub-$\sigma$-algebras $\mathcal{G} \subset \mathcal{F}$, the conditional expectation has the following properties:

- Linearity: $E(X + Y \mid \mathcal{G}) = E(X \mid \mathcal{G}) + E(Y \mid \mathcal{G})$
- Positivity: If $X \geq 0$, then $E(X \mid \mathcal{G}) \geq 0$.
- Tower property: If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ are sub-$\sigma$-algebras, then
   $$E\left(E(X \mid \mathcal{G}) \mid \mathcal{H}\right) = E(X \mid \mathcal{H})$$
\[ \mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}(X) \]

- **Factorization property:** If \( Y \) is \( \mathcal{G} \)-measurable and \( |XY| \) and \( |Y| \) are integrable, then
  \[ \mathbb{E}(XY \mid \mathcal{G}) = Y\mathbb{E}(X \mid \mathcal{G}) \]

**Proof:** Exercise.

**Definition 2.4.3 (martingale)** Let \( X_t \) be a stochastic process which is adapted to a filtration \( \{\mathcal{F}_t : t \geq 0\} \) of \( \mathcal{F} \). If

1. \( \mathbb{E}(|X_t|) < \infty \) for all \( 0 \leq t < \infty \), and
2. \( \mathbb{E}(X_t|\mathcal{F}_s) = X_s \) for all \( 0 \leq s \leq t < \infty \),

then \( X_t \) is called a **martingale**. A martingale \( X_t \) is called continuous if there is a set \( \Omega_0 \subset \Omega \) with \( \mathbb{P}(\Omega_0) = 1 \) such that the path \( t \mapsto X_t(\omega) \) is continuous for all \( \omega \in \Omega_0 \).

**Interpretation:** A martingale models a fair game. Observing the game up to time \( s \) does not give any advantage for future times.

**Examples.** It can be shown that each of the following processes is a continuous martingale with respect to the standard Brownian filtration:

\[ W_t, \quad W_t^2 - t, \quad \exp\left(\alpha W_t - \frac{\alpha^2}{2} t\right) \quad \text{with} \quad \alpha \in \mathbb{R} \]

**Proof:** Exercise.

### 2.5 Construction of the Itô integral (step 3 and 4)

**Step 3:** The Itô integral as a process

So far we have defined the Itô integral \( \mathcal{I}_T[u](\omega) \) over the interval \([0, T]\) for fixed \( T \). For applications in mathematical finance, however, we want to consider \( \{\mathcal{I}_t[u](\omega) : t \in [0, T]\} \) as a stochastic process.

If \( u(s, \omega) \in \mathcal{H}^2[0, T] \), then \( 1_{[0,t]}(s)u(s, \omega) \in \mathcal{H}^2[0, T] \). Can we define \( \mathcal{I}_t[u](\omega) \) by \( \mathcal{I}_T[1_{[0,t]}u](\omega) \)?

**Problem:** The integral \( \mathcal{I}_T[1_{[0,t]}u](\omega) \) is only defined in \( L^2(d\mathbb{P}) \). Hence, the value \( \mathcal{I}_T[1_{[0,t]}u](\omega) \) is arbitrary on sets \( Z \in \mathcal{Z}_t := \{Z \in \mathcal{F}_t : \mathbb{P}(Z) = 0\} \). Since the set \([0, T]\) is uncountable\(^2\), the union

\[ \bigcup_{t \in [0,T]} \mathcal{Z}_t \]

(i.e. the set where the process is not well-defined) could be “very large”! Fortunately, this can be fixed:

\(^2\)We only know that countable unions of null sets have measure zero, but this is not true for uncountable unions.
**Theorem 2.5.1** For any \( u \in \mathcal{H}^2[0,T] \) there is a process \( \{X_t : t \in [0,T]\} \) that is a continuous martingale with respect to the standard Brownian filtration \( \mathcal{F}_t \) such that
\[
P\left( \{ \omega \in \Omega : X_t(\omega) = \mathbb{I}_{T}[1_{[0,t]}u](\omega) \} \right) = 1
\]
for each \( t \in [0,T] \).

A proof can be found in [Ste01], Theorem 6.2, pages 83-84.

**Step 4: The Itô integral on \( \mathcal{L}^2_{\text{loc}}[0,T] \)**

So far we have defined the Itô integral for functions \( u \in \mathcal{H}^2[0,T] \); cf. Definition 2.3.1. Such functions must satisfy
\[
E \left( \int_0^T u^2(t,\omega) \, dt \right) < \infty,
\]
and this condition is sometimes too restrictive.

**Example:** If \( y(x) = \exp(x^4) \), then \( u(t,\omega) = y(W_t(\omega)) \notin \mathcal{H}^2[0,T] \).

With some more work, the Itô integral can be extended to the class \( \mathcal{L}^2_{\text{loc}}[0,T] \), i.e. to all functions
\[
u = u(t,\omega), \quad u : [0,T] \times \Omega \rightarrow \mathbb{R}
\]
with the following properties:

- \((t,\omega) \mapsto u(t,\omega)\) is \((\mathcal{B} \times \mathcal{F})\)-measurable.

- \( u \) is adapted to \( \{\mathcal{F}_t : t \in [0,T]\}\).

- \( P \left( \int_0^T u^2(t,\omega) \, dt < \infty \right) = 1 \).

The first two conditions are the same as for \( \mathcal{H}^2[0,T] \), but the third condition is weaker than \((2.9)\). If \( y : \mathbb{R} \rightarrow \mathbb{R} \) is continuous, then \( u(t,\omega) = y(W(t,\omega)) \in \mathcal{L}^2_{\text{loc}}[0,T] \), because \( t \mapsto y(W(t,\omega)) \) is continuous with probability one and hence bounded on \([0,T]\) with probability one.

**Details:** Chapter 7 in [Ste01].

**Notation**

The process \( X \) constructed above is called the **Itô integral** (Itô Kiyoshi 1944) of \( u \in \mathcal{L}^2_{\text{loc}}[0,T] \) and is denoted by
\[
X(t,\omega) = \int_0^t u(s,\omega) \, dW(s,\omega).
\]
The Itô integral over an arbitrary interval \([a, b] \subset [0, T]\) is defined by
\[
\int_a^b u(s, \omega) \, dW(s, \omega) = \int_0^b u(s, \omega) \, dW(s, \omega) - \int_0^a u(s, \omega) \, dW(s, \omega).
\]

Alternative notations:
\[
\int_a^b u(s, \omega) \, dW(s, \omega) = \int_a^b u(s, \omega) \, dW_s(\omega) = \int_a^b u_s(\omega) \, dW_s(\omega) = \int_a^b u_s \, dW_s
\]

Properties of the Itô integral

**Lemma 2.5.2** Let \(c \in \mathbb{R}\) and \(u, v \in \mathcal{L}^2_{\text{loc}}[0, T]\). The Itô integral on \([a, b] \subset [0, T]\) has the following properties:

1. **Linearity:**
\[
\int_a^b (cu(s, \omega) + v(s, \omega)) \, dW_s(\omega) = c \int_a^b u(s, \omega) \, dW_s(\omega) + \int_a^b v(s, \omega) \, dW_s(\omega)
\]
with probability one.

2. \(E\left( \int_a^b u(s, \omega) \, dW_s(\omega) \right) = 0\)

3. \(\int_a^t u(s, \omega) \, dW_s(\omega)\) is \(\mathcal{F}_t\)-measurable for \(t \geq a\).

4. **Itô isometry on \([a, b]\):**
\[
E \left( \left( \int_a^b u(s, \omega) \, dW_s(\omega) \right)^2 \right) = E \left( \int_a^b u^2(s, \omega) \, ds \right)
\]
(cf. Theorem 2.3.5).

5. **Martingale property:** The Itô integral
\[
X(t, \omega) = \int_0^t u(s, \omega) \, dW(s, \omega).
\]
of a function \(u \in \mathcal{H}^2[0, T]\) is a continuous martingale with respect to the standard Brownian filtration; cf. Theorem 2.5.1. If \(u \in \mathcal{L}^2_{\text{loc}}[0, T]\), then the Itô integral is only a local martingale; cf. Proposition 7.7 in [Ste01].

The first four properties can be shown by considering elementary functions and passing to the limit.
2.6 Stochastic differential equations and the Itô formula

Definition 2.6.1 (SDE) A stochastic differential equation (SDE) is an equation of the form

\[ X(t) = X(0) + \int_0^t f(s, X(s)) \, ds + \int_0^t g(s, X(s)) \, dW(s). \]  

(2.10)

The solution \( X(t) \) of (2.10) is called an Itô process.

The last term is an Itô integral, with \( W(t) \) denoting the Wiener process. The functions \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are called drift and diffusion coefficients, respectively. These functions are typically given while \( X(t) = X(t, \omega) \) is unknown.

This equation is actually not a differential equation, but an integral equation! Often people write

\[ dX_t = f(t, X_t)dt + g(t, X_t)dW_t \]

as a shorthand notation for (2.10). Some people even “divide by \( dt \)” in order to make the equation look like a differential equation, but this is more than audacious since “\( dW_t/dt \)” does not make sense.

Two special cases:

- If \( g(t, X(t)) \equiv 0 \), then (2.10) is reduced to

\[ X(t) = X(0) + \int_0^t f(s, X(s)) \, ds. \]

If \( X(t) \) is differentiable, this is equivalent to the ordinary differential equation

\[ \frac{dX(t)}{dt} = f(t, X(t)) \]

with initial data \( X(0) \).

- For \( f(t, X(t)) \equiv 0 \), \( g(t, X(t)) \equiv 1 \) and \( X(0) = 0 \), (2.10) turns into

\[ X(t) = X(0) + \int_0^t f(s, X(s)) \, ds + \int_0^t g(s, X(s)) \, dW(s) = W(t) - W(0) = W(t). \]
Computing Riemann integrals via the basic definition is usually very tedious. The fundamental theorem of calculus provides an alternative which is more convenient in most cases. For Itô integrals, the situation is similar: The approximation via elementary functions which is used to define the Itô integral is rarely used to compute the integral. What is the counterpart of the fundamental theorem of calculus for the Itô integral?

**Theorem 2.6.2 (Itô formula)** Let $X_t$ be the solution of the SDE

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t$$

and let $F(t, x)$ be a function with continuous partial derivatives $\partial_t F = \frac{\partial F}{\partial t}, \partial_x F = \frac{\partial F}{\partial x}$, and $\partial^2_x F = \frac{\partial^2 F}{\partial x^2}$. Then, we have for $Y_t := F(t, X_t)$ that

$$dY_t = \partial_t F \, dt + \partial_x F \, dX_t + \frac{1}{2}(\partial^2_x F)g^2 \, dt$$

$$= \left( \partial_t F + (\partial_x F)f + \frac{1}{2}(\partial^2_x F)g^2 \right) \, dt + (\partial_x F)g \, dW_t. \quad (2.11)$$

with $f = f(t, X_t), \ g = g(t, X_t), \ \partial_x F = \partial_x F(t, X_t)$, and so on.

**Notation.** Evaluations of the derivatives of $F$ are to be understood in the sense of, e.g.,

$$\partial_x F(s, X_s) := \partial_x F(t, x) \bigg|_{(t,x) = (s, X_s)}$$

and so on.

**Proof.** The proof of Theorem 2.6.2 is sketched in appendix D. A special case will be discussed in the problem class.

**Remarks:**

1. If $y(t)$ is a smooth deterministic function, then according to the chain rule the derivative of $t \mapsto F(t, y(t))$ is

$$\frac{d}{dt}F(t, y(t)) = \partial_t F(t, y(t)) + \partial_x F(t, y(t)) \cdot \frac{dy(t)}{dt}$$

and in shorthand notation

$$dF = \partial_t F \, dt + \partial_x F \, dy.$$ 

The Itô formula can be considered as a stochastic version of the chain rule, but the term $\frac{1}{2}(\partial^2_x F) \cdot g^2 \, dt$ is surprising since such a term does not appear in the deterministic chain rule.
2. Let \( f(t, X_t) = 0, g(t, X_t) = 1, X_t = W_t \) and suppose that \( F(t, x) = F(x) \) does not depend on \( t \). Then, the Itô formula yields for \( Y_t := F(W_t) \) that

\[
dY_t = F'(W_t) dW_t + \frac{1}{2} F''(W_t) dt
\]

which is the shorthand notation for

\[
F(W_t) = F(W_0) + \int_0^t F'(W_s) dW_s + \frac{1}{2} \int_0^t F''(W_s) ds.
\]

This can be seen as a counterpart of the fundamental theorem of calculus. Again, the last term is surprising, because for a suitable deterministic function \( v(t) = v_t \) we obtain

\[
F(v_t) = F(v_0) + \int_0^t F'(v_s) dv_s.
\]

**Example 1.** Consider the integral

\[
\int_0^t W_s \, dW_s.
\]

\( X_t := W_t \) solves the SDE with \( f(t, X_t) \equiv 0 \) and \( g(t, X_t) \equiv 1 \). For

\[
F(t, x) = x^2, \quad Y_t = F(t, X_t) = X_t^2 = W_t^2
\]

the Itô formula

\[
dY_t = \left( \partial_t F + (\partial_x F) f + \frac{1}{2} (\partial_x^2 F) g^2 \right) dt + (\partial_x F) g \, dW_t
\]

yields

\[
d(W_t^2) = 0 + 0 + \frac{1}{2} \cdot 2 \cdot 1^2 dt + 2W_t \cdot 1 \, dW_t = dt + 2W_t \, dW_t
\]

\[
\Rightarrow \quad W_t \, dW_t = \frac{1}{2} \left( d(W_t^2) - dt \right)
\]

This means that

\[
\int_0^t W_s \, dW_s = \frac{1}{2} \int_0^t d(W_s^2) - \frac{1}{2} \int_0^t 1 \, ds = \frac{1}{2} W_t^2 - \frac{1}{2} t.
\]

In one of the exercises, the same result was shown via approximation by elementary functions.
Example 2. The solution of the SDE
\[ dY_t = \mu Y_t \, dt + \sigma Y_t \, dW_t \]
with constants \( \mu, \sigma \in \mathbb{R} \) and deterministic initial value \( Y_0 \in \mathbb{R} \) is given by
\[ Y_t = \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right) Y_0. \]
This process is called a geometric Brownian motion and is often used in mathematical finance to model stock prices (see below).

Proof. Let \( f(t, X_t) \equiv 0 \), \( g(t, X_t) \equiv 1 \), \( X_t = W_t \) as before, but now with \( F(t, x) = \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma x \right) Y_0 \) and derivatives
\[ \partial_t F(t, x) = \left( \mu - \frac{\sigma^2}{2} \right) F(t, x), \quad \partial_i^x F(t, x) = \sigma^i F(t, x), \quad i \in \{1, 2\}. \]
Hence, the Itô formula applied to \( Y_t = F(t, X_t) = F(t, W_t) \) yields
\[
\begin{align*}
    dY_t &= \left( \left( \mu - \frac{\sigma^2}{2} \right) Y_t + \frac{1}{2} \sigma^2 Y_t \cdot 1^2 \right) dt + \sigma Y_t \cdot 1 \, dW_t \\
    &= \mu Y_t \, dt + \sigma Y_t \, dW_t.
\end{align*}
\]

Ordinary differential equations can have multiple solutions with the same initial value, and solutions do not necessarily exist for all times. Hence, we cannot expect that every SDE has a unique solution. As in the ODE case, however, existence and uniqueness can be shown under certain assumptions concerning the coefficients \( f \) and \( g \):

Theorem 2.6.3 (existence and uniqueness)

Let \( f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \) be functions with the following properties:

- **Lipschitz condition:** There is a constant \( L \geq 0 \) such that
  \[ |f(t, x) - f(t, y)| \leq L|x - y|, \quad |g(t, x) - g(t, y)| \leq L|x - y| \quad (2.12) \]
  for all \( x, y \in \mathbb{R} \) and \( t \geq 0 \).

- **Linear growth condition:** There is a constant \( K \geq 0 \) such that
  \[ |f(t, x)|^2 \leq K \left( 1 + |x|^2 \right), \quad |g(t, x)|^2 \leq K \left( 1 + |x|^2 \right) \quad (2.13) \]
  for all \( x \in \mathbb{R} \) and \( t \geq 0 \).
Then, the SDE
\[ dX(t) = f(t, X(t)) \, dt + g(t, X(t)) \, dW(t), \quad t \in [0, T] \]
with deterministic initial value \( X(0) = X_0 \) has a continuous adapted solution and
\[ \sup_{t \in [0, T]} \mathbb{E} \left( X^2(t) \right) < \infty. \]
If both \( X(t) \) and \( \tilde{X}(t) \) are such solutions, then
\[ \mathbb{P} \left( X(t) = \tilde{X}(t) \text{ for all } t \in [0, T] \right) = 1. \]

**Proof:** Theorem 9.1 in [Ste01] or Theorem 4.5.3 in [KP99].

**Remark:** The assumptions can be weakened.

### 2.7 The Feynman-Kac formula

Let \( X_t \) be the solution of the SDE
\[ dX_t = f(t, X_t) \, dt + g(t, X_t) \, dW_t, \quad t \in [t_0, T], \quad X_{t_0} = \xi \]
with suitable functions \( f \) and \( g \). Let \( u(t, x) \) be the solution of the (deterministic) partial differential equation (PDE)
\[ \partial_t u(t, x) + f(t, x) \partial_x u(t, x) + \frac{1}{2} g^2(t, x) \partial^2_x u(t, x) = 0, \quad t \in [t_0, T], \quad x \in \mathbb{R} \]
with terminal condition
\[ u(T, x) = \psi(x) \]
for some \( \psi : \mathbb{R} \rightarrow \mathbb{R} \). Apply the Itô formula (Theorem 2.6.2) to \( u(t, X_t) \):
\[ du(t, X_t) = \left( \partial_t u(t, X_t) + f(t, X_t) \partial_x u(t, X_t) + \frac{1}{2} g^2(t, X_t) \partial^2_x u(t, X_t) \right) dt + g(t, X_t) \partial_x u(t, X_t) \, dW_t \]
Equivalent:
\[ u(T, X_T) = u(t_0, X_{t_0}) + \int_{t_0}^{T} g(t, X_t) \partial_x u(t, X_t) \, dW_t \]
Taking the expectation and applying Lemma 2.5.2 yields the **Feynman-Kac formula** (Richard Feynman, Mark Kac)
\[ \mathbb{E} \left( \psi(X_T) \right) = u(t_0, \xi). \]

**Remark:** This derivation is informal, because we have tacitly assumed that all terms exist. See, e.g., Chapter 15 in [Ste01] for a correct proof.
2.8 Extension to higher dimensions

In order to model options on several underlying assets (e.g. basket options), we have to consider vector-valued Itô integrals and SDEs. A $d$-dimensional SDE takes the form

$$X_j(t) = X_j(0) + \int_0^t f_j(s, X(s)) \, ds + \sum_{k=1}^m \int_0^t g_{jk}(s, X(s)) \, dW_k(s) \quad (j = 1, \ldots, d)$$

(2.14)

for $d, m \in \mathbb{N}$ and suitable functions

$$f_j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \quad g_{jk} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}.$$  

$W_1(s), \ldots, W_m(s)$ are one-dimensional scalar Wiener processes which are pairwise independent. (2.14) is equivalent to

$$X(t) = X(0) + \int_0^t f(s, X(s)) \, ds + \int_0^t g(s, X(s)) \, dW(s)$$

(2.15)

with vectors

$$W(t) = (W_1(t), \ldots, W_m(t))^T \in \mathbb{R}^m$$

$$f(t, x) = (f_1(t, x), \ldots, f_d(t, x))^T \in \mathbb{R}^d$$

and a matrix

$$g(t, x) = \begin{pmatrix} g_{11}(t, x) & \cdots & g_{1m}(t, x) \\ \vdots & \ddots & \vdots \\ g_{d1}(t, x) & \cdots & g_{dm}(t, x) \end{pmatrix} \in \mathbb{R}^{d \times m}$$

**Theorem 2.8.1 (Multi-dimensional Itô formula) Let $X_t$ be the solution of the SDE (2.15) and let $F : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^n$ be a function with continuous partial derivatives $\partial_t F$, $\partial_{x_j} F$, and $\partial_{x_j} \partial_{x_k} F$. Then, the process $Y(t) := F(t, X_t)$ satisfies**

$$dY(t) = \partial_t F(t, X_t) \, dt$$

$$+ \sum_{i=1}^d \partial_{x_i} F(t, X_t) \cdot f_i(t, X_t) \, dt$$

$$+ \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} \partial_{x_j} F(t, X_t) \cdot \left( \sum_{k=1}^m g_{ik}(t, X_t) g_{jk}(t, X_t) \right) \, dt$$

$$+ \sum_{i=1}^d \partial_{x_i} F(t, X_t) \cdot \sum_{k=1}^m g_{ik}(t, X_t) \, dW_k$$
or equivalently

$$\frac{\partial}{\partial t} F_\ell + f^T \nabla F_\ell + \frac{1}{2} \text{tr} \left( g^T (\nabla^2 F_\ell) g \right) \ dt + (\nabla F_\ell)^T g \ dW(t)$$

where $\nabla F_\ell$ is the gradient and $\nabla^2 F_\ell$ is the Hessian of $F_\ell$, and where $\text{tr}(A) = \sum_{j=1}^{m} a_{jj}$ is the trace of a matrix $A = (a_{ij})_{i,j} \in \mathbb{R}^{m \times m}$.

**Proof:** Similar to the case $d = m = 1$.

**Final remark: Itô vs. Stratonovich.** The Itô integral is not the only stochastic integral, and the Stratonovich integral is a famous alternative. The Stratonovich integral has the advantage that the ordinary chain rule remains valid, i.e. the additional term in the Itô formula does not appear when the Stratonovich integral is used. Their disadvantage is the fact that Stratonovich integrals are not martingales, whereas Itô integrals are. Stratonovich integrals can be transformed into Itô integrals and vice versa. See 3.1, 3.3 in [Øks03] and 3.5, 4.9 in [KP99].

Actually, the SDEs (2.10) or (2.14) should be called “Itô SDE” or “SDE of the Itô type”. Since only Itô SDEs and no Stratonovich SDEs will appear in this lecture, however, we simply use the term “SDE” for “Itô SDE”.
Chapter 3

The Black-Scholes equation

References: [BK04, GJ10, Sey09]

Goal: Find equations to determine the value of an option on a single underlying asset. Throughout this chapter, we make the assumptions (A1)-(A5) from 1.3 unless otherwise stated.

3.1 Geometric Brownian motion

First step: Model the price of the underlying by a suitable process $S_t$.
For the value of a bond with interest rate $r > 0$, we have $B_t = B_0 e^{rt}$. Try to “stochastify” this equation with a Wiener process in order to model the underlying.

First attempt: $S_t = S_0 e^{at} + \sigma W_t$ for some $a, \sigma \in \mathbb{R}$. Problem: $S_t$ can have negative values. Not good.

Second attempt: $S_t = S_0 \exp(at + \sigma W_t)$ for some $a, \sigma \in \mathbb{R}$. The parameter $\sigma$ is called the volatility.
If $S_0 \geq 0$, then $S_t \geq 0$ for all $t \geq 0$. In fact, $S_t = S_0 \exp(at + \sigma W_t)$ is the geometric Brownian motion from 2.6 and solves the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with $\mu = a + \sigma^2/2$. Interpretation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

relative change = deterministic trend + random fluctuations

Lemma 3.1.1 (moments of GBM) The geometric Brownian motion

$$S_t = S_0 \exp(at + \sigma W_t), \quad a = \mu - \sigma^2/2$$

with $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and fixed (deterministic) initial value $S_0$ has the following properties:
1. $\mathbb{E}(S_t) = S_0 e^{\mu t}$
2. $\mathbb{E}(S_t^2) = S_0^2 e^{(2\mu + \sigma^2) t}$
3. $\mathbb{V}(S_t) = S_0^2 e^{2\mu t} \left( e^{\sigma^2 t} - 1 \right)$

**Proof:** Exercise.

**Definition 3.1.2 (log-normal distribution)** A vector-valued random variable $X(\omega) \in \mathbb{R}^d$ is log-normal (=log-normally distributed) if $\ln X = (\ln X_1, \ldots, \ln X_d)^T \in \mathbb{R}^d$ is normally distributed, i.e. $\ln X \sim \mathcal{N}(\xi, \Sigma)$ for some $\xi \in \mathbb{R}^d$ and a symmetric, positive definite matrix $\Sigma \in \mathbb{R}^{d \times d}$. The expectation and the covariance matrix have the entries

$$
\mathbb{E}_i(X) = e^{\xi_i + \Sigma_{ii}/2},
$$
$$
\mathbb{V}_{ij}(X) = \mathbb{E}\left( (\mathbb{E}_i(X) - X_i)(\mathbb{E}_j(X) - X_j) \right) = e^{\xi_i + \xi_j + \frac{1}{2}(\Sigma_{ii} + \Sigma_{jj}) (e^{\Sigma_{ij}} - 1)}.
$$

For $d = 1$ and $\Sigma = \sigma^2$ the corresponding density is

$$
\phi(x) = \phi(x; \xi, \sigma^2) = \begin{cases} 
\frac{1}{\sqrt{2\pi} \sigma x} \exp \left( -\frac{(\ln x - \xi)^2}{2\sigma^2} \right) & \text{if } x > 0 \\
0 & \text{else.}
\end{cases}
$$

**Proof:** Exercise.

**Example.** The (one-dimensional) geometric Brownian motion

$$
S_t = S_0 \exp(at + \sigma W_t)
$$

is log-normal, because

$$
\ln S_t = \ln S_0 + at + \sigma W_t \sim \mathcal{N}(\ln S_0 + at, \sigma^2 t)
$$

### 3.2 Derivation of the Black-Scholes equation

**Situation:** $S_t > 0$ value of an underlying, $B_t > 0$ value of a bond.

**Goal:** Determine the fair price $V_t$ of a European call or put option.

**Replication strategy:** Consider a portfolio containing $a_t \in \mathbb{R}$ underlyings and $b_t \in \mathbb{R}$ bonds such that

$$
V_t = a_t S_t + b_t B_t
$$

(cf. section 1.5). Assume that the portfolio is self-financing: no cash inflow or outflow, i.e. buying an item must be financed by selling another one. Mathematically, this means that

$$
dV_t = a_t dS_t + b_t dB_t. \tag{3.1}
$$
Interpretation: Suppose that $0 \leq r < s$ and that $a_t$ and $b_t$ are positive and piecewise constant, i.e.

$$a_t = \begin{cases} a_r & \text{if } t \in [r, s), \\ a_s & \text{if } t \geq s, \end{cases}$$

$$b_t = \begin{cases} b_r & \text{if } t \in [r, s), \\ b_s & \text{if } t \geq s. \end{cases}$$

Then, the value of the portfolio is

$$V_t = \begin{cases} a_r S_t + b_r B_t & \text{if } t \in [r, s) \\ a_s S_t + b_s B_t & \text{if } t \geq s. \end{cases}$$

If the portfolio is self-financing, then the change of $a_t$ and $b_t$ does not change the value of the portfolio, i.e. $\lim_{t \uparrow s} V_t = V_s$. This implies that

$$a_s S_s + b_s B_s = \lim_{t \uparrow s} (a_t S_t + b_t B_t) = a_r S_s + b_r B_s.$$ 

It follows that

$$V_s - V_r = (a_s S_s + b_s B_s) - (a_r S_r + b_r B_r)$$

$$= a_r (S_s - S_r) + b_r (B_s - B_r) + (a_s - a_r) S_s + (b_s - b_r) B_s = 0.$$ 

This is a discrete version of (3.1); cf. step 1 in the construction of the Itô integral.

Now we return to (3.1). Suppose that the underlying and the bond evolve according to

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

$$dB_t = r B_t dt$$

with $\mu, \sigma, r \in \mathbb{R}$. This yields

$$dV_t = a_t (\mu S_t dt + \sigma S_t dW_t) + b_t (r B_t dt)$$

$$= (a_t \mu S_t + b_t r B_t) dt + a_t \sigma S_t dW_t.$$  

(3.3)

Assume that the value of the option is a function of $t$ and $S_t$, i.e. $V_t = V(t, S_t)$. Apply the Itô formula:

$$dV(t, S_t) = \left( \partial_t V(t, S_t) + \partial_S V(t, S_t) \cdot \mu S_t + \frac{1}{2} \partial^2_{SS} V(t, S_t) \cdot \sigma^2 S_t^2 \right) dt$$

$$+ \partial_S V(t, S_t) \cdot \sigma S_t dW_t$$

(3.4)

Equating the $dW_t$-terms in (3.3) and (3.4) yields

$$a_t = \partial_S V(t, S_t),$$
while equating the $dt$-terms yields

$$a_t \mu S_t + b_t r B_t = \partial_t V(t, S_t) + \partial_S V(t, S_t) \cdot \mu S_t + \frac{1}{2} \partial_S^2 V(t, S_t) \cdot \sigma^2 S_t^2$$

$$\implies b_t r B_t = \partial_t V(t, S_t) + \frac{1}{2} \partial_S^2 V(t, S_t) \cdot \sigma^2 S_t^2$$

$$\implies b_t = \frac{1}{B_t r} \left( \partial_t V(t, S_t) + \frac{1}{2} \partial_S^2 V(t, S_t) \cdot \sigma^2 S_t^2 \right) B_t.$$

for $B_t r \neq 0$. The formulas for $a_t$ and $b_t$ yield

$$V(t, S_t) = a_t S_t + b_t B_t = \partial_S V(t, S_t) \cdot S_t + \frac{1}{B_t r} \left( \partial_t V(t, S_t) + \frac{1}{2} \partial_S^2 V(t, S_t) \cdot \sigma^2 S_t^2 \right) B_t.$$ 

Since this is true for every value of $S_t$, we can consider $S = S_t$ as a deterministic variable. Multiplying with $r$ yields the Black-Scholes equation

$$\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t, S) + r S \partial_S V(t, S) - r V(t, S) = 0.$$ 

Fischer Black and Myron Scholes 1973, Robert Merton 1973

Nobel Prize in Economics 1997

The Black-Scholes equation is a partial differential equation (PDE): It involves partial derivatives with respect to $t$ and $S$. This PDE must be solved backwards in time: instead of an initial condition, we have the terminal condition

$$V(T, S) = \psi(S)$$

where $T$ is the expiration time and $\psi(S)$ is the payoff function, i.e. $\psi(S) = (S - K)^+$ for a call and $\psi(S) = (K - S)^+$ for a put.

The Black-Scholes equation must be solved for $S \in \mathbb{R}_+ := [0, \infty)$ because only non-negative prices make sense. At the boundary $S = 0$, no boundary condition is required, because

$$\lim_{S \to 0} \frac{\sigma^2}{2} S^2 \partial_S^2 V(t, S) = 0, \quad \lim_{S \to 0} r S \partial_S V(t, S) = 0$$

if $V$ is sufficiently smooth. For $S = 0$, we obtain

$$0 = \partial_t V(t, 0) - r V(t, 0) \quad \implies \quad V(t, 0) = e^{-r(T-t)} V(T, 0). \quad (3.5)$$

This yields $V(t, 0) = 0$ for calls and $V(t, 0) = e^{-r(T-t)} K$ for puts.

**Remark.** Surprisingly, the parameter $\mu$ from (3.2) does **not** appear in the Black-Scholes equation. A similar observation has been made for the simple discrete model from 1.5.
### 3.3 Black-Scholes formulas

**First goal:** Solve the Black-Scholes equation for an European call, i.e.

\[
\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S V(t, S) + r S \partial_S V(t, S) - r V(t, S) = 0 \quad t \in [0, T], \quad S > 0 \\
V(T, S) = (S - K)^+
\]

with parameters \( r, \sigma, K, T > 0 \).

**Step 1: Transformation to the heat equation**

Define new variables:

\[
x(S) = \ln\left(\frac{S}{K}\right) \quad x : (0, \infty) \to (-\infty, \infty) \\
\tau(t) = \frac{\sigma^2}{2} (T - t) \quad \tau : [0, T] \to [0, \sigma^2 T/2] \\
w(\tau, x) = \frac{V(t, S)}{K} \quad w : [0, \sigma^2 T/2] \times (-\infty, \infty) \to \mathbb{R}
\]

Derivatives in new variables:

\[
\partial_t V(t, S) = K \partial_\tau w(\tau, x) = K \partial_\tau w(\tau, x) \frac{d\tau}{dt} = -K \frac{\sigma^2}{2} \partial_\tau w(\tau, x) \\
\partial_S V(t, S) = K \partial_x w(\tau, x) \frac{dx}{dS} = \frac{K}{S} \partial_x w(\tau, x) \\
\partial^2_S V(t, S) = \frac{K}{S^2} \left( \partial_x^2 w(\tau, x) - \partial_x w(\tau, x) \right)
\]

Insert into the Black-Scholes equation:

\[
0 = \partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S V(t, S) + r S \partial_S V(t, S) - r V(t, S) \\
= -K \frac{\sigma^2}{2} \partial_\tau w(\tau, x) + \frac{\sigma^2}{2} \frac{K}{S^2} \left( \partial_x^2 w(\tau, x) - \partial_x w(\tau, x) \right) + r \frac{K}{S} \partial_x w(\tau, x) - r K w(\tau, x)
\]

Divide by \( K \frac{\sigma^2}{2} \) and let \( c := \frac{2r}{\sigma^2} \):

\[
\partial_\tau w(\tau, x) = \partial_x^2 w(\tau, x) - \partial_x w(\tau, x) + c \partial_x w(\tau, x) - cw(\tau, x)
\]

Next, we eliminate the last three terms. Ansatz:

\[
u(\tau, x) = e^{-\alpha x - \beta \tau} w(\tau, x), \quad \alpha, \beta \in \mathbb{R}
\]

Compute \( \partial_x u(\tau, x) \):

\[
\partial_x u(\tau, x) = -\beta u(\tau, x) + e^{-\alpha x - \beta \tau} \partial_\tau w(\tau, x) \\
= -\beta u(\tau, x) + e^{-\alpha x - \beta \tau} \left( \partial_x^2 w(\tau, x) + (c - 1) \partial_x w(\tau, x) - cw(\tau, x) \right)
\]
Since
\[ \partial_x w(\tau, x) = \partial_x \left( e^{\alpha x + \beta \tau} u(\tau, x) \right) = \alpha e^{\alpha x + \beta \tau} u(\tau, x) + e^{\alpha x + \beta \tau} \partial_x u(\tau, x) \]
\[ \partial_x^2 w(\tau, x) = \alpha^2 e^{\alpha x + \beta \tau} u(\tau, x) + 2\alpha e^{\alpha x + \beta \tau} \partial_x u(\tau, x) + e^{\alpha x + \beta \tau} \partial_x^2 u(\tau, x) \]
it follows that
\[ \partial_x u(\tau, x) = -\beta u(\tau, x) + \left( \alpha^2 u(\tau, x) + 2\alpha \partial_x u(\tau, x) + \partial_x^2 u(\tau, x) \right) \]
\[ + (c - 1) \left( \alpha u(\tau, x) + \partial_x u(\tau, x) \right) - cu(\tau, x) \]
\[ = \partial_x^2 u(\tau, x) + \left( 2\alpha + (c - 1) \right) \partial_x u(\tau, x) + \left( -\beta + \alpha^2 + (c - 1)\alpha - c \right) u(\tau, x) \]
Hence, the terms including \( u(\tau, x) \) and \( \partial_x u(\tau, x) \) vanish if
\[ -\beta + \alpha^2 + (c - 1)\alpha - c = 0 \quad \text{and} \quad 2\alpha + (c - 1) = 0. \]
The solution is
\[ \alpha = -\frac{1}{2}(c - 1), \quad \beta = -\frac{1}{4}(c + 1)^2 = -(1 - \alpha)^2. \]
With these parameters, \( u(\tau, x) \) solves the heat equation
\[ \partial_x u(\tau, x) = \partial_x^2 u(\tau, x), \quad x \in \mathbb{R}, \tau \in [0, \tau_{\text{max}}] \]
with initial condition
\[ u(0, x) = e^{-\alpha x} w(0, x) = e^{-\alpha x} \frac{V(T, S)}{K} = e^{-\alpha x} \frac{(S - K)_+}{K} = e^{-\alpha x} (e^x - 1)_+ \]
since \( S = Ke^x \).

**Step 2: Solving the heat equation**

**Lemma 3.3.1 (solution of the heat equation)** Let \( u_0 : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function which satisfies the growth condition
\[ |u_0(x)| \leq Me^{\gamma x^2}. \]
with constants \( M > 0 \) and \( \gamma \geq 0 \). If \( \tau_{\text{max}} \leq 1/(4\gamma) \), then the function
\[ u(\tau, x) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} \exp \left( -\frac{(x - \xi)^2}{4\tau} \right) u_0(\xi) \, d\xi \]
is the unique solution \( u \in C([0, \tau_{\text{max}}] \times \mathbb{R}) \cap C^2((0, \tau_{\text{max}}) \times \mathbb{R}) \) of the heat equation
\[ \partial_x u(\tau, x) = \partial_x^2 u(\tau, x), \quad x \in \mathbb{R}, \tau \in (0, \tau_{\text{max}}] \]
and for all \( x \in \mathbb{R} \) we have
\[ \lim_{\tau \to 0} u(\tau, x) = u_0(x). \]
Proof: §16, 3.1., S. 407 in [FK08].

By a tedious\(^1\) calculation, it can be shown that

\[
\begin{align*}
u(\tau, x) &= \exp \left( (1 - \alpha) x + (1 - \alpha)^2 \tau \right) \Phi(d_1) - \exp \left( -\alpha x + \alpha^2 \tau \right) \Phi(d_2) \\
\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds \\
d_{1/2} &= \frac{\ln \frac{S}{K} + \left( r \pm \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}
\end{align*}
\]

(3.8)

Remark: \(\Phi(x)\) is the cumulative distribution function of the standard normal distribution.

Step 3: Inverse transform

Since \(\beta = -(1 - \alpha)^2\) and \(\beta + \alpha^2 = 2\alpha - 1 = -c\) it follows that

\[
V(t, S) = K w(\tau, x) = K \exp(\alpha x + \beta \tau) u(\tau, x)
= K \exp(\alpha x + \beta \tau) \exp \left( (1 - \alpha) x + (1 - \alpha)^2 \tau \right) \Phi(d_1) \\
- K \exp(\alpha x + \beta \tau) \exp \left( -\alpha x + \alpha^2 \tau \right) \Phi(d_2)
= K \exp(x) \Phi(d_1) - K \exp \left( \beta + \alpha^2 \tau \right) \Phi(d_2)
= S \Phi(d_1) - K \exp(-r(T - t)) \Phi(d_2)
\]

Check behaviour on the boundary: Since \(\lim_{S \to 0} \Phi(d_{1/2}(S)) = 0\) we obtain

\[
\lim_{S \to 0} V(t, S) = \lim_{S \to 0} \left[ S \Phi(d_1(S)) - Ke^{-r(T-t)} \Phi(d_2(S)) \right] = 0 \iff (3.5) \checkmark
\]

Check terminal condition:

\[
V(T, S) = S \Phi(d_1) - K \Phi(d_2)
\]

By definition of \(d_{1/2} = d_{1/2}(t)\)

\[
\lim_{t \to T} d_{1/2}(t) = \lim_{t \to T} \frac{\ln \frac{S}{K} + \left( r \pm \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}} = \lim_{t \to T} \frac{\ln \frac{S}{K}}{\sigma \sqrt{T - t}} = \begin{cases} 
\infty & \text{if } S > K \\
0 & \text{if } S = K \\
-\infty & \text{if } S < K
\end{cases}
\]

\(^{1}\) so tedious that we do not even dare to ask the reader to prove this as an exercise.
and hence
\[
\lim_{t \to T} \Phi(d_{1/2}(t)) = \begin{cases} 
1 & \text{if } S > K \\
1/2 & \text{if } S = K \\
0 & \text{if } S < K
\end{cases}
\implies \lim_{t \to T} V(t, S) = \begin{cases} 
S - K & \text{if } S > K \\
0 & \text{if } S = K \\
0 & \text{if } S < K.
\end{cases}
\]

All in all, we have shown the following:

**Theorem 3.3.2 (Black-Scholes formula for calls)** If \( r, \sigma, K, T > 0 \), then the **Black-Scholes formula**
\[
V(t, S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)
\]
with \( \Phi \) and \( d_{1/2} \) from (3.8) and (3.9), respectively, is the unique solution of the Black-Scholes equation for European calls, i.e.
\[
\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t, S) + rS \partial_S V(t, S) - rV(t, S) = 0 \quad t \in [0, T], \quad S > 0
\]
\[
V(T, S) = (S - K)^+.
\]

**Corollary 3.3.3 (Black-Scholes formula for puts)** The Black-Scholes equation for a European put
\[
\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t, S) + rS \partial_S V(t, S) - rV(t, S) = 0 \quad t \in [0, T], \quad S > 0
\]
\[
V(T, S) = (K - S)^+
\]
with \( r, \sigma, K, T > 0 \) has the unique solution
\[
V(t, S) = Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1)
\]
with \( \Phi \) and \( d_{1/2} \) from (3.8) and (3.9), respectively.

**Proof.** Let \( V_C(t, S) \) be the value of a call with the same \( T \) and \( K \). The put-call-parity (Lemma 1.4.1) and Theorem 3.3.2 imply
\[
V(t, S) = e^{-r(T-t)}K + V_C(t, S) - S
\]
\[
= e^{-r(T-t)}K + S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) - S
\]
\[
= e^{-r(T-t)}K(1 - \Phi(d_2)) + S(\Phi(d_1) - 1)
\]
\[
= e^{-r(T-t)}K\Phi(-d_2) - S\Phi(-d_1)
\]
because \( \Phi(x) + \Phi(-x) = 1 \).
Definition 3.3.4 (Greeks) For a European option with value \( V(t, S) \) we define “the greeks”

- **delta:** \( \Delta = \partial_S V \)
- **gamma:** \( \Gamma = \partial^2_S V \)
- **theta:** \( \theta = \partial_t V \)
- **rho:** \( \rho = \partial_r V \)
- **vega/kappa:** \( \kappa = \partial_\sigma V \)

These partial derivatives can be considered as “condition numbers” which measure the sensitivity of \( V(t, S) \) with respect to the corresponding parameters. This information is important for stock broker.

**Remark:** Explicit formulas for the greeks can be derived from the Black-Scholes formulas (exercise).

### 3.4 Risk-neutral valuation and equivalent martingale measures

In 1.5 we have seen that in the simplified two-scenario model the value of an option can be priced by replication. The same strategy was applied to the refined model in the previous section. In the simple situation considered in 1.5, the value of an option turned out to be the discounted expectation of the payoff under the risk-neutral probability. In this subsection, we will see that this is also true for the refined model from 3.2.

**Theorem 3.4.1 (Option price as a discounted expectation)** If \( V(t, S) \) is the solution of the Black-Scholes equation

\[
\begin{align*}
\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S V(t, S) + rS \partial_S V(t, S) - rV(t, S) &= 0 \quad t \in [0, T], \quad S > 0 \\
V(T, S) &= \psi(S)
\end{align*}
\]

with payoff function \( \psi(S) \), then

\[
V(t_*, S_*) = e^{-r(T-t_*)} \int_0^\infty \psi(x) \phi(x, \xi, \beta^2) \, dx \quad (3.10)
\]
for all \( t_* \in [0, T] \) and \( S_* > 0 \). The function \( \phi \) is the density of the log-normal distribution (cf. Definition 3.1.2) with parameters

\[ \xi = \ln S_* + \left( r - \frac{\sigma^2}{2} \right) (T - t_*), \quad \beta^2 = \sigma^2 (T - t_*) \]  

(3.11)

The assertion can be shown by showing that the above representation coincides with the Black-Scholes formulas for puts and calls. Such a proof, however, involves several changes of variables in the integral representations and rather tedious calculations. We give a shorter and more elegant proof:

**Proof. Step 1:** In our derivation of the Black-Scholes model, we have assumed that

\[ dS_t = \mu S_t dt + \sigma S_t dW_t, \]

i.e. that the price of the underlying is a geometric Brownian motion with drift \( \mu S_t \); cf. (3.2). It turned out, however, that the parameter \( \mu \) does not appear in the Black-Scholes equation. Hence, we can choose \( \mu = r \) and consider the SDE

\[ d\hat{S}_t = r \hat{S}_t dt + \sigma \hat{S}_t dW_t, \quad t \in [t_*, T] \]

\[ \hat{S}_{t_*} = S_* \]

as a model for the stock price.

**Step 2:** The function \( u(t, S) := e^{r(T-t)} V(t, S) \) solves the PDE

\[ \partial_t u(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S u(t, S) + r S \partial_S u(t, S) = 0, \quad t \in [0, T] \]

because

\[
\begin{align*}
\partial_t u(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S u(t, S) + r S \partial_S u(t, S) &= -re^{r(T-t)} V(t, S) + e^{r(T-t)} \partial_t V(t, S) + \frac{\sigma^2}{2} S^2 e^{r(T-t)} \partial^2_S V(t, S) + r S e^{r(T-t)} \partial_S V(t, S) \\
= e^{r(T-t)} \left( -r V(t, S) + \partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S V(t, S) + r S \partial_S V(t, S) \right) &= 0.
\end{align*}
\]

Moreover, \( u \) satisfies the terminal condition

\[ u(T, S) = V(T, S) = \psi(S). \]

**Step 3:** Applying the Feynman-Kac formula (cf. Section 2.7) with \( f(t, S) = r S \) and \( g(t, S) = \sigma S \) yields

\[ E \left( \psi(\hat{S}_T) \right) = u(t_*, S_*) = e^{r(T-t_*)} V(t_*, S_*) \]
and thus
\[ V(t_*, S_*) = e^{-r(T-t_*)} \mathbb{E} \left( \psi(\hat{S}_T) \right). \]

We know that \( \hat{S}_T \) is log-normal, i.e.
\[ \mathbb{E} \left( \psi(\hat{S}_T) \right) = \int_0^\infty \psi(x) \phi(x, \xi, \beta^2) \, dx \]
where \( \phi(x, \xi, \beta) \) is the density of the log-normal distribution with parameters (3.11); cf. the example after Definition 3.1.2.

**Interpretation.** We know from Definition 3.1.2 that
\[ \mathbb{E} \left( \hat{S}_T \right) = \int_0^\infty x \phi(x, \xi, \beta^2) \, dx = \exp \left( \xi + \frac{\beta^2}{2} \right) \]
\[ = \exp \left( \ln S_* + \left( r - \frac{\sigma^2}{2} \right) (T - t_*) + \frac{1}{2} \sigma^2 (T - t_*) \right) \]
\[ = \exp \left( \ln S_* + r (T - t_*) \right) \]
\[ = S_* \exp \left( r (T - t_*) \right). \]
This means that for \( \mu = r \) the expected value of the stock is exactly the money obtained by investing \( S_* \) into a bond at time \( t_* \) and waiting until \( T \). Hence, the log-normal distribution with parameters (3.11) defines the **risk-neutral probability**; cf. 1.5. The integral in (3.10) is precisely the expected payoff under the risk-neutral probability, and (3.10) states that the price of the option is obtained by discounting the expected payoff.

**A different perspective.** Consider now the geometric Brownian motion
\[ dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \]
with \( \mu \neq r \). Since \( \mathbb{E}(S_t) = S_0 e^{\mu t} \), an investor expects that \( \mu > r \) as a compensation for the risk, because otherwise he might prefer to invest into the riskless bond \( B_t = B_0 e^{rt} \). The term
\[ \gamma = \frac{\mu - r}{\sigma} \]
is called **market price of risk**, and we have
\[ dS_t = r S_t dt + \sigma S_t (\gamma dt + dW_t) = r S_t dt + \sigma S_t dW_t^\gamma, \]
with \( W_t^\gamma = \gamma t + W_t \).

**Problem:** \( W_t^\gamma \) is not a Wiener process under the probability measure \( \mathbb{P} \), because
\[ \mathbb{E}(W_t^\gamma) = \gamma t + \mathbb{E}(W_t) = \gamma t \neq 0 \]
for \( t > 0 \) and \( \mu \neq r \).

**Question:** Is there another probability measure \( \mathbb{Q} \) such that \( W_t^\gamma \) is a Wiener process under \( \mathbb{Q} \)?

**Definition 3.4.2 (equivalent martingale measure)** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \( \{\mathcal{F}_t : t \geq 0\} \). A probability measure \( \mathbb{Q} \) is called an **equivalent martingale measure** or **risk-neutral probability** if there is a random variable \( Y > 0 \) such that
\[ \cdot \mathbb{Q}(A) = \mathbb{E}(1_A \cdot Y) = \int_A Y(\omega) d\mathbb{P}(\omega) \]
for all events \( A \in \mathcal{F} \), and
\[ \cdot e^{-rt} S_t \]
is a martingale under \( \mathbb{Q} \) with respect to the filtration \( \{\mathcal{F}_t : t \geq 0\} \).

**Remark.** The first property implies that \( \mathbb{P}(A) > 0 \iff \mathbb{Q}(A) > 0 \) (“equivalent”).

Now let
\[ Y_T := \exp \left( -\gamma W_T - \frac{\gamma^2}{2} T \right) \]
and \( \mathbb{Q}(A) = \mathbb{E}(1_A \cdot Y_T) \).

Then, Girsanov’s theorem states that \( W_t^\gamma = \gamma t + W_t \) is a Wiener process under \( \mathbb{Q} \) (see e.g. 4.4 in [Ben04], 8.6 in [Øks03]). Moreover, the Itô formula yields
\[ d(e^{-rt} S_t) = -re^{-rt} S_t dt + e^{-rt} \left( r S_t dt + \sigma S_t dW_t^\gamma \right) = \sigma e^{-rt} S_t dW_t^\gamma. \]
Hence, \( e^{-rt} S_t \) is a martingale under \( \mathbb{Q} \), and \( \mathbb{Q} \) is an equivalent martingale measure. All in all, we have exchanged
\[ \mu \rightarrow r, \quad \mathbb{P} \rightarrow \mathbb{Q}, \quad W_t \rightarrow W_t^\gamma. \]

Now we are back in the situation of Theorem 3.4.1, and it follows that
\[ V(t_*, S_*) = e^{-r(T-t_*)} \mathbb{E}_\mathbb{Q}(\psi(S_T)) \]
(3.12)
where \( S_t \) is the solution of
\[ dS_t = r S_t dt + \sigma S_t dW_t^\gamma, \quad t \in [t_*, T] \]
\[ S_{t_*} = S_* \]
If \( \mu = r \), then \( \mathbb{P} = \mathbb{Q} \) and \( W_t = W_t^\gamma \).

**General pricing formula.** Up to now, we have only considered European options, i.e. options with a payoff that depends only on the value of the underlying at maturity. For Asian or barrier options, the pricing formula (3.12) can be generalized to
\[ V_t = e^{-r(T-t)} \mathbb{E}_\mathbb{Q}(V_T | \mathcal{F}_t) \]
(cf. 5.2.4 in [Shr04]).
Fundamental theorems of option pricing:

- If a market model has at least one equivalent martingale measure, then there is no arbitrage possibility (cf. Theorem 5.4.7 in [Shr04]).
- Consider a market model with at least one equivalent martingale measure. Then, the equivalent martingale measure is unique if and only if the model is complete, i.e. if every derivative (options, forwards, futures, swaps, ...) can be replicated (hedged) (cf. Theorem 5.4.9 in [Shr04]).

3.5 Extensions

The “standard” Black-Scholes model can be generalized in several ways:

- Options with $d > 1$ underlyings (e.g. basket options) are modeled by the $d$-dimensional Black-Scholes equation

$$
\partial_t V + \frac{1}{2} \sum_{i,j=1}^{d} \rho_{ij} \sigma_i \sigma_j S_i S_j \partial_{S_i} \partial_{S_j} V + r \sum_{i=1}^{d} S_i \partial_{S_i} V - r V = 0
$$

where $V = V(t, S_1, \ldots, S_d)$, $r, \sigma_i > 0$ and $\rho_{ij} \in [-1, 1]$ are the correlation coefficients.

- Non-constant interest rate and volatility: $r = r(t, S), \sigma = \sigma(t, S)$

- Stochastic volatility: Either $\sigma = \sigma(\omega)$ is a random variable with known distribution or $\sigma = \sigma_t(\omega)$ is a stochastic process.

- Dividends: When a dividend $\delta \cdot S_t$ with $\delta \geq 0$ is paid at time $t$, the price of the underlying drops by the same amount due to the no-arbitrage assumption. Hence, a continuous flow of dividends can be modeled by

$$
dS_t = (\mu - \delta)S_t dt + \sigma S_t dW_t,
$$

which yields the Black-Scholes equation

$$
\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S V(t, S) + (r - \delta) S \partial_S V(t, S) - rV(t, S) = 0.
$$

Black-Scholes formulas with dividends: 4.5.1 in [GJ10].

- Nonzero transaction costs $\Rightarrow$ nonlinear Black-Scholes equation

- Discontinuous underlyings: Jump-diffusion models, Black-Scholes PDE with additional integral term

Remark: Some of these extensions will be considered in the lecture Numerical methods in mathematical finance II (summer term).
Chapter 4

Binomial methods

**Situation:** Let $S(t)$ be the value of an underlying, and let $V(t, S)$ be the value of an option with maturity $T > 0$.

**Assumptions:** Assume (A1)-(A5) from Section 1.3.

**Goal:** Approximate $V(t, S)$ by a numerical method.

**Idea:** Refine the simple discrete model from 1.5 such that it approximates the continuous-time Black-Scholes model.

**Remark:** For European calls/puts, such an approximation is not necessary, because $V(t, S)$ can be computed via the Black-Scholes formula. Nevertheless, such options will serve as a model problem. The numerical method can be extended to other types of options.

### 4.1 Derivation

Discretize the time-interval $[0, T]$: Choose $N \in \mathbb{N}$, let $\tau = T/N$ and $t_n = n \cdot \tau$.

Let $S_n$ be the price of the underlying at time $t_n$. Bond: $B(t) = B(0)e^{rt}$

**Additional assumptions:**

1. For a given price $S_n$, the price at $t_{n+1} = t_n + \tau$ is

   $$S_{n+1} = \begin{cases} u \cdot S_n & \text{with probability } p \\ d \cdot S_n & \text{with probability } 1 - p \end{cases}$$

   with (unknown) $u > 1$, $0 < d < 1$, $p \in [0, 1]$.

2. The expected profit from investing into the underlying is the same as for the bond:

   $$\mathbb{E}(S_{n+1}|S_n) = e^{r\tau} S_n.$$

3. $\mathbb{E}(S_{n+1}^2|S_n) = e^{(2r+\sigma^2)\tau} S_n^2$ with given volatility $\sigma \in \mathbb{R}$.

**Remark:** In the continuous-time model where $S(t)$ is modeled by a geometric Brownian motion, the last two conditions hold for $S_n = S(t_n)$ if $\mu = r$ (risk-neutral pricing).
Compute \( u, d, p \):

1. \( e^{r\tau} S_n = \mathbb{E}(S_{n+1} | S_n) = uS_n p + dS_n (1 - p) \iff p = \frac{e^{r\tau} - d}{u - d} \)

   Since \( p \in [0, 1] \), it follows that \( d \leq e^{r\tau} \leq u \).

2. \( e^{(2r+\sigma^2)\tau} S_n^2 = \mathbb{E}(S_{n+1}^2 | S_n) = (uS_n)^2 p + (dS_n)^2 (1 - p) \)

   Only two conditions for three unknowns \( u, d, p \). Choose third condition:

3. \( u \cdot d = 1 \)

Solution of 1.-3.:

\[
\begin{align*}
  u &= \beta + \sqrt{\beta^2 - 1} \\
  d &= \frac{1}{u} = \beta - \sqrt{\beta^2 - 1} \\
  p &= \frac{e^{r\tau} - d}{u - d} \\
\end{align*}
\]

(4.1)

Replication strategy: Consider a portfolio containing \( a \in \mathbb{R} \) underlyings and \( b \in \mathbb{R} \) bonds such that

\[
a S_n + b B(t_n) \overset{\dagger}{=} V(t_n, S_n) =: V_n
\]

It follows that

\[
\begin{align*}
  \mathbb{E}(V_{n+1} | V_n) &= a \mathbb{E}(S_{n+1} | S_n) + b e^{r\tau} B(t_n) \\
  &= e^{r\tau} (a S_n + b B(t_n)) \\
  &= e^{r\tau} V_n \\
\end{align*}
\]

(4.2)

4.2 Algorithm

Cox, Ross & Rubinstein 1979

1. **Forward phase: initialization of the tree.** For all \( n = 0, \ldots, N \) and \( j = 0, \ldots, n \) let

   \[
   S_{jn} = u^j d^{n-j} S(0) = \text{(approximate) price of the underlying at time } t_n \text{ after } j \text{ “ups” and } n - j \text{ “downs”}.
   \]

The condition \( d \cdot u = 1 \) implies that

\[
\begin{align*}
  S(0) &= S_{00} = S_{12} = S_{24} = \ldots \\
  S_{11} &= S_{23} = S_{35} = \ldots \\
  S_{01} &= S_{13} = S_{25} = \ldots
\end{align*}
\]

At \( t_n \) there are only \( n + 1 \) possible values \( S_{0n}, \ldots, S_{nn} \) of the underlying.
\[ S_{00} = S(0) \]
\[ \text{for } n = 0, 1, 2, \ldots, N - 1 \]
\[ S_{0,n+1} = dS_{0,n} \]
\[ \text{for } j = 0, \ldots, n \]
\[ S_{j+1,n+1} = uS_{j,n} \]
\[ \text{end} \]
\[ \text{end} \]

2. Backward phase: compute the option values. Let \( V_{jn} \) be the value of the option after \( j \) “ups” and \( n - j \) “downs” of the underlying. At maturity, we have

\[ V_{jN} = \psi(S_{jN}), \quad \psi(S_{jN}) = \begin{cases} (S_{jN} - K)^+ & \text{(call)} \\ (K - S_{jN})^+ & \text{(put)} \end{cases} \]

Use (4.2):

\[ e^{\tau r} V_{jn} = \mathbb{E}(V(t_{n+1}) \mid V_{jn}) = pV_{j+1,n+1} + (1 - p)V_{j,n+1} \]

\[ \implies V_{jn} = e^{-\tau r} \left( pV_{j+1,n+1} + (1 - p)V_{j,n+1} \right) \]

(a) European options:

\[ \text{for } j = 0, \ldots, N \]
\[ V_{jN} = \psi(S_{jN}) \]
\[ \text{end} \]
\[ \text{for } n = N - 1, N - 2, \ldots, 0 \]
\[ \text{for } j = 0, \ldots, n \]
\[ V_{jn} = e^{-\tau r} \left( pV_{j+1,n+1} + (1 - p)V_{j,n+1} \right) \]
\[ \text{end} \]
\[ \text{end} \]
Result: \( V_{00} \)

(b) American options: Check in each step if early exercise is advantageous. At time \( t_n \) the value of the option must not be less than \( \psi(S_{jn}) \) for all \( j \).
for $j = 0, \ldots, N$
\[ V_{jN} = \psi(S_{jN}) \]
end
for $n = N - 1, N - 2, \ldots, 0$
    for $j = 0, \ldots, n$
        $\tilde{V}_{jn} = e^{-r\tau}(pV_{j+1,n+1} + (1 - p)V_{j,n+1})$
        $V_{jn} = \max\{\tilde{V}_{jn}, \psi(S_{jn})\}$
    end
end
Result: $V_{00}$

Remarks.
1. The result $V_{00}$ is only an approximation for the true value $V(0, S(0))$ of the option, because the price process has been approximated.
2. The result $V_{00} \approx V(0, S(0))$ depends on the initial value $S(0)$. For a different value of $S(0)$, the entire computation must be repeated.
3. An efficient implementation of the binomial method requires only $O(N)$ operations; see [Hig02].
4.3 Discrete Black-Scholes formula

Lemma 4.3.1 Let $V(t, S)$ be the value of a European option with payoff function $\psi(S)$ and maturity $T > 0$. Then, the binomial method yields the approximation

$$V_0 = e^{-rT} \sum_{j=0}^{N} B(j, N, p) \psi(S_{jN})$$

where

$$B(j, N, p) = \binom{N}{j} p^j (1 - p)^{N-j}$$

is the binomial distribution.

Proof: exercise.

Remarks:
1. This result explains the name “binomial method”.
2. Interpretation: $V_0$ is the discounted expected payoff under a suitable probability; cf. 1.5 and 3.4.

Proposition 4.3.2 (Discrete Black-Scholes formula) For a European call with maturity $T$ and strike $K$, the binomial method yields the approximation

$$V_0 = S(0) \Psi(m, N, q) - Ke^{-rT} \Psi(m, N, p)$$

where

$$q = up e^{-r\tau}$$

$$m = \min\{0 \leq j \leq N : (S_{jN} - K) \geq 0\}.$$ 

$$\Psi(m, N, p) = \sum_{j=m}^{N} B(j, N, p)$$

Proof: exercise.

Question: What happens if we let $N \rightarrow \infty$ and $\tau = T/N \rightarrow 0$?

For simplicity, we consider a slightly different binomial method. For given $\sigma > 0$ let

$$\hat{u} = \hat{u}(\tau) = e^{\sigma \sqrt{\tau}} \quad \text{and} \quad \hat{d} = \hat{d}(\tau) = \frac{1}{\hat{u}} = e^{-\sigma \sqrt{\tau}} \quad (4.3)$$

and suppose that $u$ and $d$ from (4.1) are replaced by $\hat{u}$ and $\hat{d}$. It can be shown, however, that

$$\hat{u}(\tau) - u = O(\tau^{3/2}). \quad (4.4)$$

If $\tau \leq (\sigma/r)^2$, then we have $\hat{d} \leq e^{r\tau} \leq \hat{u}$ as before.
Proposition 4.3.3 (Convergence of the discrete Black-Scholes formula)
Consider a European call with maturity $T$ and strike $K$. Let $V_{00}^{(N)}$ be the approximation given by the binomial method with $\tau = T/N$, and for $u$ and $d$ replaced by $\hat{u}$ and $\hat{d}$, respectively. Then, $V_{00}^{(N)}$ converges to the value given by the (continuous) Black-Scholes formula:

$$\lim_{N \to \infty} V_{00}^{(N)} = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds$$

$$d_{1/2} = \ln\left(\frac{S(0)}{K}\right) + \left( r \pm \frac{\sigma^2}{2}\right) T \sigma \sqrt{T}$$

Proof: See 3.3 in [GJ10] (use central limit theorem).
Chapter 5

Numerical methods for stochastic differential equations

5.1 Motivation

According to 3.4 the value of a European option is the discounted expected payoff under the risk-neutral probability:

\[ V(0, S_0) = e^{-rT} \mathbb{E}_Q \left( \psi(S(T)) \right) \]

For the standard Black-Scholes model:

\[ V(0, S_0) = e^{-rT} \int_0^\infty \psi(x) \phi(x, \xi, \beta^2) \, dx \]

with log-normal density \( \phi \) and parameters

\[ \xi = \ln S_0 + \left( r - \frac{\sigma^2}{2} \right) T, \quad \beta = \sigma \sqrt{T}. \]

Two ways to price the option:

1. **Quadrature formula.** Let \( w(x) := \psi(x) \phi(x, \xi, \beta^2) \).
   Choose \( 0 \leq x_{\min} < x_{\max} \) such that \( w(x) \approx 0 \) for \( x \notin [x_{\min}, x_{\max}] \).
   \( x_{\min} = K \) and \( x_{\max} \) sufficiently large for calls, \( x_{\min} = 0 \) and \( x_{\max} = K \) for puts.
   Choose large \( N \in \mathbb{N} \), let \( h = (x_{\max} - x_{\min})/N \) and \( x_k = x_{\min} + kh \). Approximate

   \[ \int_0^\infty w(x) \, dx \approx \int_{x_{\min}}^{x_{\max}} w(x) \, dx = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} w(x) \, dx \approx \sum_{k=0}^{N-1} h \sum_{j=1}^s b_j w(x_k + c_j h) \]

   with suitable nodes \( c_j \in [0, 1] \) and weights \( b_j \).
2. **Monte Carlo method.** In the Black-Scholes model, $S(t)$ is defined by the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dB(t), \quad t \in [0, T], \quad S_0 \text{ given}$$

(risk-neutral, $\mu = r$)

Solution: Geometric Brownian motion

$$S(t) = S_0 \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma B(t) \right).$$

This is the process which corresponds to $\phi(x, \xi, \beta^2)$, because $S(T)$ is log-normal with the same parameters. Estimate the expected payoff as follows:

- Generate many realizations $S(T,\omega_1), \ldots, S(T,\omega_m), m \in \mathbb{N}$ “large”.
- Approximate

$$V(0, S_0) \approx e^{-rT} \frac{1}{m} \sum_{j=1}^{m} \psi(S(T, \omega_j)).$$

Consider now a more complicated price process:

$$dS(t) = rS(t)dt + \sigma(t)S(t)dB(t) \quad (5.1a)$$

$$d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \nu \left( \rho dB_1(t) + \sqrt{1-\rho^2} dB_2(t) \right) \quad (5.1b)$$

Heston model with parameters $r, \kappa, \theta, \nu > 0$, initial values $S_0, \sigma_0$, independent scalar Wiener processes $B_1(t), B_2(t)$, correlation $\rho \in [-1, 1]$

Steven L. Heston 1993

Now the volatility is not a parameter, but a stochastic process defined by a second SDE. We do not have an explicit formula for $S(t)$ and $\sigma(t)$, but the Monte Carlo approach is still feasible:

- Choose $N \in \mathbb{N}$, define step-size $\tau = T/N$ and $t_n = n\tau$. For each $\omega_1, \ldots, \omega_m$ compute approximations

$$X^{(1)}(\omega_j) \approx S(t_n, \omega_j), \quad X^{(2)}(\omega_j) \approx \sigma^2(t_n, \omega_j), \quad n = 0, \ldots, N$$

by solving the SDEs (5.1a), (5.1b) numerically.
- Approximate

$$V(0, S_0) \approx e^{-rT} \frac{1}{m} \sum_{j=1}^{m} \psi(X^{(1)}(\omega_j)) \approx S(T, \omega_j)$$

The Monte Carlo approach even works for other types of options. As an example, consider an Asian option with payoff

$$\psi(t \mapsto S(t)) = \left( S(T) - \frac{1}{T} \int_0^T S(t) \, dt \right)^+ \quad \text{(average strike call)}.$$
Now the payoff depends on the entire path $t \mapsto S(t)$. We approximate

$$S(t_n, \omega_j) \approx X_n^{(1)}(\omega_j), \quad \frac{1}{T} \int_0^T S(t, \omega_j) \, dt \approx \frac{1}{N} \sum_{n=1}^N X_n^{(1)}(\omega_j)$$

and hence

$$V(0,S_0) \approx e^{-rT} \frac{1}{m} \sum_{j=1}^m \left( X_N^{(1)}(\omega_j) - \frac{1}{N} \sum_{n=1}^N X_n^{(1)}(\omega_j) \right)$$

**Remark:** In the original paper, Heston derives an explicit Black-Scholes-type formula for European options by means of characteristic functions. Hence, European options in the Heston model can also be priced by quadrature formulas, but for Asian options this is impossible.

**Goal:** Construct and analyze numerical methods for SDEs.

### 5.2 Euler-Maruyama method

#### 5.2.1 Derivation

Consider the one-dimensional SDE

$$dX(t) = f(t, X(t)) \, dt + g(t, X(t)) \, dW(t), \quad t \in [0, T], \quad X(0) = X_0$$

with suitable functions $f$ and $g$ and a given initial value $X_0$. Choose $N \in \mathbb{N}$, define step-size $\tau = T/N$ and $t_n = n\tau$.

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} f(s, X(s)) \, ds + \int_{t_n}^{t_{n+1}} g(s, X(s)) \, dW(s)$$

$$\approx X(t_n) + (t_{n+1} - t_n) f(t_n, X(t_n)) + g(t_n, X(t_n)) \left( W(t_{n+1}) - W(t_n) \right) =: \Delta W_n$$

Replacing $X(t_n) \rightarrow X_n$ and “$\approx$” $\rightarrow$ “$=$” yields the

**Euler-Maruyama method** (Gisiro Maruyama 1955, Leonhard Euler 1768-70): For $n = 0, \ldots, N - 1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

$$X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n) \Delta W_n.$$

If $g(t, x) \equiv 0$, then the Euler-Maruyama method reduces to the explicit Euler method applied to the ordinary differential equation $X'(t) = f(t, X(t))$. 
Hope that $X_n \approx X(t_n)$:

<table>
<thead>
<tr>
<th>SDE</th>
<th>recursion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(t)$ exact</td>
<td>$\implies$ approx.</td>
</tr>
<tr>
<td>$X_n$ approx.</td>
<td>$\Leftarrow$ exact</td>
</tr>
</tbody>
</table>

The exact solution $X(t_n)$ and the numerical approximation $X_n$ are random variables. For every path $t \mapsto W(t, \omega)$ of the Wiener process, a different result is obtained. $X(t)$ is called strong solution if $t \mapsto W(t, \omega)$ is given, and weak solution if $t \mapsto W(t, \omega)$ can be chosen. Approximations of weak solutions: For each $n$, generate a random number $Z_n \sim \mathcal{N}(0, 1)$ and let

$$\Delta W_n = \sqrt{\tau} Z_n.$$

Question: Does $X_n$ really approximate $X(t_n)$? In which sense? How accurately?

### 5.2.2 Weak and strong convergence

**Definition 5.2.1 (strong and weak convergence)** Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$ and $t_n = n \tau$. An approximation $X_n(\omega) \approx X(t_n, \omega)$ converges

- **strongly** with order $\gamma > 0$, if there is a constant $C > 0$ independent of $\tau$ such that

$$\max_{n=0, \ldots, N} \mathbb{E}(|X(t_n) - X_n|) \leq C \tau^\gamma$$

for all sufficiently small $\tau$, and

- **weakly** with order $\gamma > 0$ with respect to a function $F : \mathbb{R} \to \mathbb{R}$, if there is a constant $C > 0$ independent of $\tau$ such that

$$\max_{n=0, \ldots, N} \left| \mathbb{E}[F(X(t_n))] - \mathbb{E}[F(X_n)] \right| \leq C \tau^\gamma$$

for all sufficiently small $\tau$.

**Remarks:**

- Strong convergence $\implies$ path-wise convergence
  Weak convergence $\implies$ convergence of moments (if $F(x) = x^k$) or probabilities (if $F(x) = 1_{[a,b]}$).
- Strong convergence of order $\gamma$ implies weak convergence of order $\geq \gamma$ with respect to $F(x) = x$ (exercise).
5.2.3 Strong convergence of the Euler-Maruyama method

For simplicity, we only consider the autonomous SDE
\[ dX(t) = f(X(t)) \, dt + g(X(t)) \, dW(t), \quad t \in [0,T] \]
and the Euler-Maruyama approximation
\[ X_{n+1} = X_n + \tau f(X_n) + g(X_n) \Delta W_n \]
with \( X(0) = X_0, \, T > 0, \, N \in \mathbb{N}, \, \tau = T/N, \, t_n = n\tau. \)

We assume that \( f = f(x) \) and \( g = g(x) \) satisfy the Lipschitz condition (2.12): There is a constant \( L \geq 0 \) such that
\[
|f(x) - f(y)| \leq L|x - y|, \quad |g(x) - g(y)| \leq L|x - y| \tag{2.12}
\]
for all \( x, y \in \mathbb{R} \). In the autonomous case, this implies the linear growth condition (2.13) (exercise).

**Theorem 5.2.2 (strong error of the Euler-Maruyama method)** Under these conditions, there is a constant \( \hat{C} \) such that
\[
\max_{n=0,\ldots,N} \mathbb{E}\left(|X(t_n) - X_n|\right) \leq \hat{C} \tau^{1/2}
\]
for all sufficiently small \( \tau \). \( \hat{C} \) does not depend on \( \tau \).

For the proof we need the following

**Lemma 5.2.3 (Gronwall)** Let \( \alpha : [0,T] \rightarrow \mathbb{R}_+ \) be a positive integrable function. If there are constants \( a > 0 \) and \( b > 0 \) such that
\[
0 \leq \alpha(t) \leq a + b \int_0^t \alpha(s) \, ds
\]
for all \( t \in [0,T] \), then \( \alpha(t) \leq ae^{bt} \).

**Proof:** exercise.

**Proof of Theorem 5.2.2.**

**Strategy:**
- Define the step function
\[
Y(t) = \sum_{n=0}^{N-1} 1_{[t_n,t_{n+1})}(t)X_n \quad \text{for } t \in [0,T), \quad Y(T) := X_N.
\]
For every \( n = 0, \ldots, N - 1 \) this means that \( Y(t) = X_n \) for all \( t \in [t_n, t_{n+1}) \).
• Define $\alpha(s) := \sup_{r \in [0,s]} \mathbb{E}(|Y(r) - X(r)|^2)$ and prove the Gronwall inequality

$$0 \leq \alpha(t) \leq C\tau + b \int_0^t \alpha(s) \, ds. \tag{5.2}$$

• Apply Gronwall’s lemma. This yields $\alpha(t) \leq \hat{\tau}C^2$ with $\hat{\tau}C^2 = Ce^{bT}$ for all $t \in [0,T]$

• Since $\mathbb{E}(Z) \leq \sqrt{\mathbb{E}(Z^2)}$ for random variables $Z$, it follows that

$$\max_{n=0,\ldots,N} \mathbb{E}(|X_n - X(t_n)|) \leq \sup_{t \in [0,T]} \mathbb{E}(|Y(t) - X(t)|) \leq \sup_{t \in [0,T]} \sqrt{\mathbb{E}(|Y(t) - X(t)|^2)} = \sqrt{\alpha(T)} \leq \sqrt{\hat{\tau}C}$$

Main challenge: Prove Gronwall inequality (5.2). Choose fixed $t \in [0,T]$ and let $n$ be the index with $t \in [t_n, t_{n+1})$.

Derive integral representation of the error:

$$Y(t) = X_n = X_0 + \sum_{k=0}^{n-1} (X_{k+1} - X_k)$$

$$= X_0 + \sum_{k=0}^{n-1} \left( \tau f(X_k) + g(X_k) \Delta W_k \right)$$

$$= X_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(X_k) \, ds + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} g(X_k) \, dW(s)$$

$$= X_0 + \int_0^{t_n} f(Y(s)) \, ds + \int_0^{t_n} g(Y(s)) \, dW(s)$$

Comparing with the exact solution

$$X(t) = X(0) + \int_0^t f(X(s)) \, ds + \int_0^t g(X(s)) \, dW(s)$$

---

1Elementary calculation: $0 \leq V(Z) = \mathbb{E}[(Z - \mathbb{E}(Z))^2] = \mathbb{E}[Z^2 - 2Z\mathbb{E}(Z) + \mathbb{E}(Z)^2] = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2$ and hence $(\mathbb{E}(Z))^2 \leq \mathbb{E}(Z^2)$. 
yields the error representation

\[ Y(t) - X(t) = \int_0^t \left[ f(Y(s)) - f(X(s)) \right] ds + \int_0^t \left[ g(Y(s)) - g(X(s)) \right] dW(s) \]

\[ =: T_1 \]

\[ - \int_{t_n}^t f(X(s)) ds - \int_{t_n}^t g(X(s)) dW(s) \]

\[ =: T_2 \]

\[ = T_1 + T_2 - T_3 - T_4. \]

Setting \( T = (T_1, T_2, T_3, T_4)^T \) and applying the Cauchy-Schwarz inequality gives

\[ (T_1 + T_2 - T_3 - T_4)^2 \leq (1, 1, -1, -1)T^2 \leq 4\|T\|^2 = 4 \cdot (T_1^2 + T_2^2 + T_3^2 + T_4^2) \]

and hence

\[ \mathbb{E}(|Y(t) - X(t)|^2) \leq 4 \cdot \mathbb{E}(T_1^2 + T_2^2 + T_3^2 + T_4^2). \]

**First term:** For functions \( u \in L^2([0, t_n]) \) the Cauchy-Schwarz inequality yields

\[ \left( \int_0^{t_n} u(s) \cdot 1 \, ds \right)^2 \leq \int_0^{t_n} |u(s)|^2 \, ds \cdot \int_0^{t_n} 1^2 \, ds. \quad (5.3) \]

Using the Lipschitz bound (2.12), we obtain

\[ \mathbb{E}(T_1^2) = \mathbb{E} \left[ \left( \int_0^{t_n} \left[ f(Y(s)) - f(X(s)) \right] ds \right)^2 \right] \]

\[ \leq t_n \mathbb{E} \left( \int_0^{t_n} \left| f(Y(s)) - f(X(s)) \right|^2 ds \right) \]

\[ \leq TL^2 \int_0^{t_n} \mathbb{E} \left( |Y(s) - X(s)|^2 \right) ds \]

\[ \leq TL^2 \int_0^t \alpha(s) \, ds \quad (t \text{ instead of } t_n) \]
because $t \geq t_n$ by assumption.

**Second term:** It follows from the Itô isometry (Theorem 2.3.5) and the Lipschitz bound (2.12) that

$$
\mathbb{E} \left( \mathcal{T}_2^2 \right) = \mathbb{E} \left[ \left( \int_{0}^{t_n} \left[ g(Y(s)) - g(X(s)) \right] dW(s) \right)^2 \right] \\
= \mathbb{E} \left( \int_{0}^{t_n} \left| g(Y(s)) - g(X(s)) \right|^2 ds \right) \\
\leq L^2 \int_{0}^{t_n} \mathbb{E} \left( \left| Y(s) - X(s) \right|^2 \right) ds \\
\leq L^2 \int_{0}^{t} \alpha(s) ds
$$

because $t \geq t_n$ by assumption.

**Third term:** Equation (5.3) and the linear growth bound (2.13) yield

$$
\mathbb{E} \left( \mathcal{T}_3^2 \right) = \mathbb{E} \left[ \left( \int_{t_n}^{t} f(X(s)) ds \right)^2 \right] \\
\leq (t - t_n) \mathbb{E} \left( \int_{t_n}^{t} \left| f(X(s)) \right|^2 ds \right) \\
\leq \tau K \cdot \mathbb{E} \left( \int_{t_n}^{t} \left( 1 + |X(s)|^2 \right) ds \right) \leq c \tau^2
$$

because Theorem 2.6.3 states that $\mathbb{E} (1 + |X(s)|^2)$ remains bounded on $[t_n, t]$. 

Last term: Using the Itô isometry and the linear growth bound (2.13) it follows that

$$\mathbb{E}(T^2) = \mathbb{E}\left[\left(\int_{t_n}^t g(X(s))dW(s)\right)^2\right]$$

$$= \mathbb{E}\left(\int_{t_n}^t |g(X(s)|^2 ds\right)$$

$$\leq K \cdot \mathbb{E}\left(\int_{t_n}^t (1 + |X(s)|^2) ds\right) \leq c\tau$$

These bounds yield the Gronwall inequality (5.2) with $b = 4(T + 1)L^2$ and with $C$ depending on $K$ and $\sup_{s \in [0,T]} \mathbb{E}(1 + |X(s)|^2)$.

5.2.4 Weak convergence of the Euler-Maruyama method

Theorem 5.2.4 (weak error of the Euler-Maruyama method) Under the conditions of 5.2.3, there is a constant $\hat{C}$ such that

$$\max_{n=0,\ldots,N} \left| \mathbb{E}[F(X(t_n))] - \mathbb{E}[F(X_n)] \right| \leq \hat{C}\tau$$

for all sufficiently small $\tau$ and all sufficiently smooth functions $F$. $\hat{C}$ does not depend on $\tau$.

Proof. Define interpolation: In addition to the piecewise constant $Y(t)$, we define the interpolation

$$Z(t) = X_n + (t - t_n)f(X_n) + g(X_n)(W(t) - W(t_n)) \quad \text{for } t \in [t_n, t_{n+1}).$$

Properties:

- $Z(t_n) = X_n$ for all $n = 0, \ldots, N$.
- For every $\delta \in [0, \tau]$, $Z(t_n + \delta)$ is the Euler-Maruyama approximation after one step with step-size $\delta$ and initial value $Z(t_n) = X_n$.
- $t \mapsto Z(t, \omega)$ is continuous with probability 1.
- Since $Y(t) = X_n$ for $t \in [t_n, t_{n+1})$, this is equivalent to

$$Z(t) = X(0) + \int_0^t f(Y(s))ds + \int_0^t g(Y(s))dW(s)$$

or

$$dZ = f(Y)dt + g(Y)dW(t).$$
Choose \( n \in \{1, \ldots, N\} \) and consider the error at time \( t_n \).

**Apply the Feynman-Kac formula:** Let \( u(t, x) \) be the solution of the PDE

\[
\partial_t u(t, x) + f(x) \partial_x u(t, x) + \frac{1}{2} g^2(x) \partial_x^2 u(t, x) = 0, \quad t \in [0, t_n], \quad x \in \mathbb{R}
\]

with terminal condition \( u(t_n, x) = F(x) \). Apply the Itô formula to \( u(t, Z(t)) \):

\[
du(t, Z) = \left( \frac{\partial_t u(t, Z) + f(Y) \partial_x u(t, Z) + \frac{1}{2} g^2(Y) \partial_x^2 u(t, Z)}{\partial_x u(t, Z)} \right) dt + g(Y) \partial_x u(t, Z) dW(t)
\]

Equivalent:

\[
u(t_n, Z(t_n)) = u(0, Z(0)) + \int_0^{t_n} \left[ f(Y) - f(Z) \right] \partial_x u(t, Z) dt
\]

\[
+ \frac{1}{2} \int_0^{t_n} \left[ g^2(Y) - g^2(Z) \right] \partial_x^2 u(t, Z) dt
\]

\[
+ \int_0^{t_n} g(Y) \partial_x u(t, Z) dW(t)
\]

By construction: \( u(t_n, Z(t_n)) = u(t_n, X_n) = F(X_n) \)

Feynman-Kac (see 2.7): \( u(0, Z(0)) = u(0, X(0)) = \mathbb{E}[F(X(t_n))] \)

This yields

\[
\left| \mathbb{E}[F(X_n)] - \mathbb{E}[F(X(t_n))] \right| \leq \int_0^{t_n} \left| \mathbb{E}\left[ \left( f(Y) - f(Z) \right] \partial_x u(t, Z) \right) \right| dt
\]

\[
\leq T_1 + \frac{1}{2} \int_0^{t_n} \left| \mathbb{E}\left( \left[ g^2(Y) - g^2(Z) \right] \partial_x^2 u(t, Z) \right) \right| dt
\]

**Bounds for \( T_1 \) and \( T_2 \):** Define

\[
G(t, x) = \left[ f(Y(t)) - f(x) \right] \partial_x u(t, x)
\]
and apply the Itô formula to $G(t, Z)$:

$$dG(t, Z) = \left( \partial_t G(t, Z) + \partial_x G(t, Z) \cdot f(Y) + \frac{1}{2} \partial_x^2 G(t, Z) \cdot g^2(Y) \right) dt + \partial_x G(t, Z) \cdot g(Y) dW(t)$$

Equivalent:

$$G(t, Z(t)) = G(t_{n-1}, Z(t_{n-1})) + \int_{t_{n-1}}^{t} \partial_t G(s, Z) \, ds + \int_{t_{n-1}}^{t} \partial_x G(s, Z) \cdot f(Y) \, ds$$

$$+ \frac{1}{2} \int_{t_{n-1}}^{t} \partial_x^2 G(t, Z) \cdot g^2(Y) \, ds + \int_{t_{n-1}}^{t} \partial_x G(s, Z) \cdot g(Y) \, dW(s)$$

where $Z = Z(s)$ and $Y = Y(s)$. Consider $\mathbb{E}(\ldots)$:

$$\mathbb{E}[G(t, Z(t))] = \int_{t_{n-1}}^{t} \mathbb{E}\left( \partial_t G(s, Z) \right) \, ds + \int_{t_{n-1}}^{t} \mathbb{E}\left( \partial_x G(s, Z) \cdot f(Y) \right) \, ds$$

$$+ \frac{1}{2} \int_{t_{n-1}}^{t} \mathbb{E}\left( \partial_x^2 G(t, Z) \cdot g^2(Y) \right) \, ds + 0$$

It can be shown that all three integrands remain bounded. It follows that

$$|\mathbb{E}[G(t, Z(t))]| \leq C \cdot (t - t_{n-1}) \leq C T.$$

Consequence:

$$\mathcal{T}_1 = \int_{0}^{t_n} \left| \mathbb{E}\left( [f(Y) - f(Z)] \partial_x u(t, Z) \right) \right| \, dt = \int_{0}^{t_n} \left| \mathbb{E}\left[ G(t, Z(t)) \right] \right| \, dt \leq C t_n \tau \leq C T \tau.$$

In a similar way, it can be shown that $\mathcal{T}_2 \leq C t_n \tau \leq C T \tau$. This proves that

$$\left| \mathbb{E}[F(X(t_n))] - \mathbb{E}[F(X_n)] \right| \leq \hat{C} \tau.$$

#### 5.3 Higher-order methods

Consider again the one-dimensional SDE

$$dX(t) = f(X(t)) \, dt + g(X(t)) \, dW(t), \quad t \in [0, T], \quad X(0) = X_0$$

with suitable functions $f$ and $g$ and a given initial value $X_0$.

**Goal:** Construct numerical methods with higher order.
Caution! Numerical methods for ordinary differential equations can in general not be extended to stochastic differential equations!

Example: Heun’s method.

Heun’s method for the ODE \( \dot{y}(t) = f(y) \) with initial value \( y(0) = y_0 \) takes the form

\[
\tilde{y}_{n+1} = y_n + \tau f(y_n),
\]

\[
y_{n+1} = y_n + \frac{\tau}{2} \left( f(y_n) + f(\tilde{y}_{n+1}) \right).
\]

Similar to trapezoidal rule, but explicit. The natural modification of this method for SDEs is

\[
\tilde{X}_{n+1} = X_n + \tau f(X_n) + g(X_n)\Delta W_n,
\]

\[
X_{n+1} = X_n + \frac{\tau}{2} \left( f(X_n) + f(\tilde{X}_{n+1}) \right) + \frac{1}{2} \left( g(X_n) + g(\tilde{X}_{n+1}) \right) \Delta W_n.
\]

Consider the special case \( f(x) \equiv 0, g(x) = x \). For the exact solution \( X(t) \), it follows that

\[
E(X(t)) = E(X_0) + \int_0^t E\left[ f(X(s)) \right] ds + E\left[ \int_0^t g(X(s))dW(s) \right],
\]

i.e. that \( E(X(t)) \) is constant. The method simplifies to

\[
\tilde{X}_{n+1} = X_n + X_n\Delta W_n,
\]

\[
X_{n+1} = X_n + \frac{1}{2} \left( X_n + \tilde{X}_{n+1} \right) \Delta W_n.
\]

or equivalently

\[
X_{n+1} = X_n + X_n\Delta W_n + \frac{1}{2} X_n(\Delta W_n)^2.
\]

This yields

\[
E(X_{n+1}) = E(X_n) + E(X_n\Delta W_n) + \frac{1}{2} E \left( X_n(\Delta W_n)^2 \right) = E(X_n) + \frac{\tau}{2} E(X_n)
\]

and thus for \( N \rightarrow \infty \) and \( \tau = T/N \rightarrow 0 \)

\[
\lim_{\tau \to 0} E(X_N) = \lim_{\tau \to 0}(1 + \tau/2)^N X_0 = \lim_{N \rightarrow \infty} \left( 1 + \frac{T}{2N} \right)^N X_0 = e^{T/2} X_0.
\]

Hence, the method is not consistent! No convergence!
**Stochastic Taylor expansions**

Important tool for the construction of higher-order methods. For smooth $F : \mathbb{R} \rightarrow \mathbb{R}$, the Itô formula yields

$$dF(X) = \left( F'(X) \cdot f(X) + \frac{1}{2} F''(X) \cdot g^2(X) \right) dt + F'(X) \cdot g(X) dW(t)$$

(no time derivative, because $F = F(x)$ does not depend on $t$) or equivalently

$$F(X(s)) = F(X(t_n)) + \int_{t_n}^{s} \mathcal{L}_0 F(X(r)) \, dr + \int_{t_n}^{s} \mathcal{L}_1 F(X(r)) \, dW(r) \tag{5.4}$$

Choose $F(x) = f(x)$ and $F(x) = g(x)$, respectively, and substitute into the SDE:

$$X(t) = X(t_n) + \int_{t_n}^{t} f(X(s)) \, ds + \int_{t_n}^{t} g(X(s)) \, dW(s)$$

$$= X(t_n) + \int_{t_n}^{t} f(X(t_n)) \, ds + \int_{t_n}^{s} \mathcal{L}_0 f(X(r)) \, dr \, ds + \int_{t_n}^{s} \mathcal{L}_1 f(X(r)) \, dW(r) \, ds$$

$$+ \int_{t_n}^{t} g(X(t_n)) \, dW(s) + \int_{t_n}^{s} \mathcal{L}_0 g(X(r)) \, dr \, dW(s) + \int_{t_n}^{s} \mathcal{L}_1 g(X(r)) \, dW(r) \, dW(s)$$

$$= \underbrace{(t-t_n)f(X(t_n))}_{T_{11}} + \underbrace{\int_{t_n}^{t} \mathcal{L}_0 f(X(r)) \, dr \, ds}_{T_{12}}$$

$$+ \underbrace{g(X(t_n)) [W(t) - W(t_n)]}_{T_{21}} + \underbrace{\int_{t_n}^{s} \mathcal{L}_0 g(X(r)) \, dr \, dW(s) + \int_{t_n}^{s} \mathcal{L}_1 g(X(r)) \, dW(r) \, dW(s)}_{T_{22}}$$

with

$$\mathcal{L}_0 f = f' \cdot f + \frac{1}{2} f'' \cdot g^2 \quad \mathcal{L}_1 f = f' \cdot g$$

$$\mathcal{L}_0 g = g' \cdot f + \frac{1}{2} g'' \cdot g^2 \quad \mathcal{L}_1 g = g' \cdot g$$

If all double integrals $T_{ij}$ are ignored and if $t = t_{n+1}$, then we obtain the Euler-Maruyama method.

**Higher strong order**

Since

$$\mathbb{E} \left[ (W(t_{n+1}) - W(t_n))^2 \right] = t_{n+1} - t_n,$$
we conjecture that for $t \to t_n$ the dominant integral term is

$$T_{22} = \int_t^s \int_{t_n}^{t_n} L_1 g(X(r)) \, dW(r) dW(s).$$

Ignoring $T_{11}, T_{12}, T_{21}$ yields the approximation

$$X(t) \approx X(t_n) + (t - t_n) f(X(t_n)) + g(X(t_n))(W(t) - W(t_n)) + T_{22}.$$ 

In order to approximate $T_{22}$, we apply (5.4) with $F(x) = L_1 g(x)$ and ignore higher-order terms:

$$T_{22} = \int_t^s \int_{t_n}^{t_n} L_1 g(X(r)) \, dW(r) dW(s) 
\approx L_1 g(X(t_n)) \int_t^s dW(r) dW(s) 
= g'(X(t_n)) \cdot g(X(t_n)) \int_t^s dW(r) dW(s)$$ 

(5.6)

The integral can be explicitly computed:

$$\int_t^s dW(r) dW(s) = \int_t^{t_n} [W(s) - W(t_n)] dW(s)$$ 

$$= \int_t^{t_n} W(s) dW(s) - W(t_n) \int_t^{t_n} dW(s)$$ 

$$= \int_0^{t_n} W(s) dW(s) - \int_0^{t_n} W(s) dW(s) - W(t_n) [W(t) - W(t_n)]$$ 

$$= \frac{1}{2} (W^2(t) - t) - \frac{1}{2} (W^2(t_n) - t_n) - W(t_n) [W(t) - W(t_n)]$$ 

$$= \frac{1}{2} W^2(t) + \left(1 - \frac{1}{2}\right) W^2(t_n) - W(t_n) W(t) + \frac{1}{2} (t_n - t)$$ 

$$= \frac{1}{2} [W(t) - W(t_n)]^2 - \frac{1}{2} (t - t_n)$$

For $t = t_{n+1} = t_n + \tau$, this yields the 

**Milstein method** (Grigori N. Milstein 1974):
For \( n = 0, \ldots, N - 1 \) let \( \Delta W_n = W(t_{n+1}) - W(t_n) \) and

\[
X_{n+1} = X_n + \tau f(X_n) + g(X_n) \Delta W_n + g'(X_n) \cdot g(X_n) \frac{1}{2} \left( (\Delta W_n)^2 - \tau \right).
\]

Strong order 1, weak order 1 (cf. 10.3 in [KP99])

**Remark:** Milstein = Euler-Maruyama + additional term

If \( g'(x) = 0 \) ("additive noise") then Milstein = Euler-Maruyama

**Problem:** Have to compute derivative \( g' \). Difficult if \( g \) is complicated or not explicitly given (i.e. no formula for \( g \)).

**Idea:** Avoid \( g' \) by using a (sufficiently accurate) approximation. Let

\[
\tilde{X}_{n+1} = X_n + \tau f(X_n) + \sqrt{\tau} g(X_n).
\]

(similar to Euler-Maruyama, but with \( \sqrt{\tau} \) instead of \( \Delta W_n \)). Taylor yields

\[
g(\tilde{X}_{n+1}) = g(X_n) + g'(X_n) \left[ \tilde{X}_{n+1} - X_n \right] + \mathcal{O}\left( \| \tilde{X}_{n+1} - X_n \|^2 \right)
\]

\[
= g(X_n) + g'(X_n) \left[ \tau f(X_n) + g(X_n) \sqrt{\tau} \right] + \mathcal{O}(\tau)
\]

and hence

\[
g'(X_n) g(X_n) = \frac{g(\tilde{X}_{n+1}) - g(X_n)}{\sqrt{\tau}} + \mathcal{O}(\sqrt{\tau}).
\]

This yields the

**Stochastic Milstein-Runge-Kutta method:**

For \( n = 0, \ldots, N - 1 \) let \( \Delta W_n = W(t_{n+1}) - W(t_n) \) and

\[
\tilde{X}_{n+1} = X_n + \tau f(X_n) + \sqrt{\tau} g(X_n)
\]

\[
X_{n+1} = X_n + \tau f(X_n) + g(X_n) \Delta W_n + \frac{g(\tilde{X}_{n+1}) - g(X_n)}{\sqrt{\tau}} \cdot \frac{1}{2} \left[ (\Delta W_n)^2 - \tau \right].
\]

Strong order 1, weak order 1 (in spite of the additional approximation).

**Higher weak order**

Go back to (5.5), i.e.

\[
X(t) = X(t_n) + (t - t_n) f(X(t_n)) + \mathcal{T}_{11} + \mathcal{T}_{12}
\]

\[
+ g(X(t_n)) \left[ W(t) - W(t_n) \right] + \mathcal{T}_{21} + \mathcal{T}_{22}
\]
and approximate each double integral by freezing the integrand at $t_n$:

\[ T_{11} = \int_{t_n}^{t} \int_{t_n}^{s} L_0 f(X(r)) \, dr \, ds \approx \int_{t_n}^{t} \int_{t_n}^{s} L_0 f(X(t_n)) \, dr \, ds = \frac{(t - t_n)^2}{2} L_0 f(X(t_n)) \]

\[ T_{12} = \int_{t_n}^{t} \int_{t_n}^{s} L_1 f(X(r)) \, dW(r) \, ds \approx L_1 f(X(t_n)) \mathcal{I}_{(1,0)}(t) \]

\[ T_{21} = \int_{t_n}^{t} \int_{t_n}^{s} L_0 g(X(r)) \, dr \, dW(s) \approx L_0 g(X(t_n)) \mathcal{I}_{(0,1)}(t) \]

\[ T_{22} \approx L_1 g(X(t_n)) \left( [W(t) - W(t_n)]^2 - (t - t_n) \right) \quad \text{(cf. (5.6))} \]

with

\[ \mathcal{I}_{(1,0)}(t) = \int_{t_n}^{t} \int_{t_n}^{s} dW(r) \, ds = \int_{t_n}^{t} W(s) - W(t_n) \, ds \]

\[ \mathcal{I}_{(0,1)}(t) = \int_{t_n}^{t} \int_{t_n}^{s} dr \, dW(s) = \int_{t_n}^{t} (s - t_n) \, dW(s). \]

Applying the integration by parts formula

\[ \int_{a}^{b} u(s) \, dW(s) = u(b)W(b) - u(a)W(a) - \int_{a}^{b} W(s)u'(s) \, ds \quad (5.7) \]

(see exercises) with $a = t_n$, $b = t$, $u(s) = s - t_n$ gives

\[ I_{(0,1)}(t) = \int_{t_n}^{t} (s - t_n) \, dW(s) \]

\[ = (t - t_n)W(t) - 0 - \int_{t_n}^{t} W(s) \cdot 1 \, ds \]

\[ = (t - t_n)W(t) - W(t_n) - \int_{t_n}^{t} [W(s) - W(t_n)] \, ds \]

\[ = (t - t_n)W(t) - W(t_n) - \mathcal{I}_{(1,0)}(t) \]

Hence, only $\mathcal{I}_{(1,0)}(t)$ has to be computed. For weak convergence all random variables can be replaced by other random variables with the same moments. It can be shown (exercise) that

\[ \mathbb{E}(\mathcal{I}_{(1,0)}(t)) = 0, \quad \mathbb{E}(\mathcal{I}_{(1,0)}^2(t)) = \frac{(t - t_n)^3}{3}, \quad \mathbb{E}\left(\mathcal{I}_{(1,0)}(t)[W(t) - W(t_n)]\right) = \frac{(t - t_n)^2}{2}. \]
Let $Z_1 \sim \mathcal{N}(0, 1)$ and $Z_2 \sim \mathcal{N}(0, 1)$ be independent and $W(t) - W(t_n) = \sqrt{t - t_n}Z_1$. If we let

$$Y_n := \frac{\tau^{3/2}}{2} \left( Z_1 + \frac{1}{\sqrt{3}}Z_2 \right),$$

then $Y_n$ has the same properties as $\mathcal{I}_{(1,0)}(t_{n+1})$, i.e.

$$\mathbb{E}(Y_n) = 0, \quad \mathbb{E}(Y_n^2) = \frac{\tau^3}{3}, \quad \mathbb{E}(Y_n\Delta W_n) = \frac{\tau^2}{2}$$

(exercise). For $t = t_{n+1}$, this yields the following method:

$$X_{n+1} = X_n + \tau f(X_n) + \frac{\tau^2}{2} \mathcal{L}_0 f(X_n) + \mathcal{L}_1 f(X_n)Y_n \approx T_{t_1}
+ \tau g(X_n) \Delta W_n + \mathcal{L}_0 g(X_n) \left[ \tau \Delta W_n - Y_n \right] \approx T_{t_2}
+ \mathcal{L}_1 g(X_n) \frac{1}{2} \left( \Delta W_n^2 - \tau \right) \approx T_{t_3}$$

Weak order 2 (no proof).

**Simplification:** The weak order is not reduced if $\Delta W_n$ is replaced by a “cheaper” random variable with the same moments. Let $\Delta V_n \in \{\sqrt{3}\tau, -\sqrt{3}\tau, 0\}$ with

$$\mathbb{P}(\Delta V_n = \sqrt{3}\tau) = \mathbb{P}(\Delta V_n = -\sqrt{3}\tau) = \frac{1}{6}, \quad \mathbb{P}(\Delta V_n = 0) = \frac{2}{3}.$$  

It can be checked (exercise) that

$$\mathbb{E}\left((\Delta V_n)^k\right) = \mathbb{E}\left((\Delta W_n)^k\right) \quad \text{for } k = 1, \ldots, 5.$$  

Moreover, we can replace $Y_n$ by

$$\tilde{Y}_n = \frac{\tau}{2} \Delta V_n,$$

because

$$\mathbb{E}(\tilde{Y}_n) = 0, \quad \mathbb{E}(\tilde{Y}_n^2) = \frac{\tau^3}{4} = \frac{\tau^3}{3} + \mathcal{O}(\tau^3), \quad \mathbb{E}(\tilde{Y}_n\Delta V_n) = \frac{\tau^2}{2}.$$  

### 5.4 Numerical methods for systems of SDEs

Consider now the vector-valued SDE

$$X(t) = X(0) + \int_0^t f(s, X(s)) \, ds + \int_0^t g(s, X(s)) \, dW(s)$$
with

\[ X(t) \in \mathbb{R}^d, \quad W(t) \in \mathbb{R}^m, \quad f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad \text{and} \quad g : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}. \]

Notation as in 2.8.

The Euler-Maruyama method can be readily extended to vector-valued SDEs: For \( n = 0, \ldots, N - 1 \) let \( \Delta W_n = W(t_{n+1}) - W(t_n) \) and

\[ X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n) \Delta W_n. \]

Strong order 1/2, weak order 1, similar proof.

What about the Milstein method?

• Case \( d \geq 1 \) and \( m = 1 \): Straightforward extension.

For \( n = 0, \ldots, N - 1 \) let \( \Delta W_n = W(t_{n+1}) - W(t_n) \) and

\[ X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n) \Delta W_n + J_g(t_n, X_n) g(t_n, X_n) \frac{1}{2} \left( \Delta W_n^2 - \tau \right) \]

where \( J_g = [\partial_{x_k} g_j]_{j,k} \) is the Jacobian. Strong order 1, weak order 1.

• Case \( d \geq 1 \) and \( m > 1 \): More complicated. Adapting the derivation via stochastic Taylor expansions yields

\[
X_{n+1}^{(j)} = X_n^{(j)} + \tau f_j(t_n, X_n) + \sum_{k=1}^{m} g_{jk}(t_n, X_n) \Delta W_n^{(k)} + \sum_{i,k=1}^{d} \sum_{l=1}^{d} \partial_{x_l} g_{jk}(t_n, X_n) \cdot g_{li}(t_n, X_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_i(\theta) dW_k(s)
\]

where \( X_n^{(j)} \) and \( \Delta W_n^{(j)} \) denote the \( j \)-th entry of \( X_n \) and \( \Delta W_n \), respectively. Similar to the scalar case, the derivatives of \( g \) can be avoided by a Runge-Kutta-type approach.

The stochastic integrals cannot be computed analytically. These integrals are solutions of small systems of SDEs, which have to be approximated numerically. Details: 5.3 in [GJ10].

### 5.5 Mean-square-error of the Monte Carlo simulation

Consider a European option with payoff function \( \psi \) and price process

\[ dS(t) = f(t, S(t)) dt + g(t, S(t)) dW(t), \quad t \in [0, T]. \tag{5.8} \]

Standard Monte Carlo method (cf. section 5.1):
Choose $N \in \mathbb{N}$, let $\tau = T/N$ and $t_n = n\tau$ for $n = 0, \ldots, N$.

Generate $m \in \mathbb{N}$ paths $t \mapsto W(t, \omega_j)$ of the Wiener process ($j = 1, \ldots, m$). For each path, compute approximations $S_n(w_j) \approx S(t_n, \omega_j)$ by solving (5.8) with a numerical method of weak order $\gamma$.

Approximate the discounted expectation:

$$E_Q \left[ \psi(S(T)) \right] \approx \frac{1}{m} \sum_{j=1}^{m} \psi(S(T, \omega_j)) \approx \frac{1}{m} \sum_{j=1}^{m} \psi(S_N(\omega_j)) =: \hat{V}$$

Two sources of error:

- Estimate the expectation from finitely many samples.
- Approximate the exact $S(T, \omega_j)$ by a numerical method.

Both errors are measured by the mean-square-error.

**Definition 5.5.1 (Mean-square-error)** Let $\hat{\theta}$ be an estimator for an unknown (deterministic) quantity $\theta$. Then, the mean-square-error of $\hat{\theta}$ is

$$MSE(\hat{\theta}) = E \left[ (\hat{\theta} - \theta)^2 \right] \overset{(*)}{=} V(\hat{\theta}) + E(\hat{\theta} - \theta)^2.$$ 

The term $E(\hat{\theta} - \theta) = E(\hat{\theta}) - \theta$ is called the bias. Notation: $E(X)^2 := (E(X))^2 \neq E(X^2)$.

Proof of $(*)$:

$$V(\hat{\theta}) + E(\hat{\theta} - \theta)^2 = \left( E(\hat{\theta}^2) - E(\hat{\theta})^2 \right) + \left( E(\hat{\theta}^2) - 2E(\hat{\theta})\theta + \theta^2 \right)$$

$$= E\left( \hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2 \right) = E\left[ (\hat{\theta} - \theta)^2 \right].$$

Applying this with $E = E_Q$ and

$$\theta = E\left( \psi(S(T)) \right), \quad \hat{\theta} = \hat{V} = \frac{1}{m} \sum_{j=1}^{m} \psi(S_N(\omega_j))$$

yields

$$MSE(\hat{V}) = \frac{1}{m^2} V\left( \sum_{j=1}^{m} \psi(S_N(\omega_j)) \right) + \left( \frac{1}{m} \sum_{j=1}^{m} E\left[ \psi(S_N(\omega_j)) \right] - E\left[ \psi(S(T)) \right] \right)^2$$

$$= \frac{V(\psi(S_N))}{m} + \left( E[\psi(S_N)] - E[\psi(S(T))] \right)^2$$

$$\leq \frac{C}{m} + C\tau^{2\gamma}.$$ 

Consequence: $\sqrt{MSE(\hat{V})} \sim C\sqrt{m^{-1} + \tau^{2\gamma}}$.

Slow convergence with respect to $m$!
Example: Euler-Maruyama method. If $\varepsilon > 0$ is a given error tolerance, then

$$\text{MSE}(\hat{V}) = \varepsilon^2 \leq \frac{C}{m} + C\tau^2 \iff m = \mathcal{O}(\varepsilon^{-2}) \quad \text{and} \quad \tau = \mathcal{O}(\varepsilon).$$

Since $\tau = T/N$, we have to compute $m = \mathcal{O}(\varepsilon^{-2})$ simulations with $N = \mathcal{O}(\varepsilon^{-1})$ time-steps. Hence, the total numerical work (= total number of time-steps) is $\mathcal{O}(\varepsilon^{-3})$.

The computational costs can be reduced by Multi-Level Monte Carlo methods, cf. part 2 of the lecture.
Chapter 6

Pseudo-random numbers and Monte Carlo integration

6.1 Pseudo-random numbers

Stochastic simulations are based on random variables. In order to approximate the weak solution of a SDE, for example, the Wiener increment $\Delta W_n = W(t_{n+1}) - W(t_n)$ is simulated by drawing a random number $Z_n \sim N(0,1)$ and letting $\Delta W_n = \sqrt{\tau}Z_n$; cf. 5.2.

**Question:** What does it mean to “draw a random number”? How can a computer generate a random number?

Computers can only generate pseudo-random numbers, i.e. sequences of numbers which seem to be random, but which are actually generated by a deterministic algorithm. Hence, simulation results can be reproduced if necessary. Every such sequence is periodic, but with a very large period.

6.1.1 Uniform pseudo-random numbers

First goal: generate uniformly distributed pseudo-random numbers $X_i \in [0,1]$.

Notation: $X_i \sim U(0,1)$.

MATLAB command: `rand(...)`

**Method 1: Linear congruential generator**

Choose $M \in \mathbb{N}$ and $a, b, X_0 \in \{0,1,\ldots,M-1\}$ and let

\[ X_i = (aX_{i-1} + b) \mod M, \quad U_i = \frac{X_i}{M} \quad (i = 1, 2, 3, \ldots) \]

Reminder: $x \mod y = z \iff x = ny + z$ for some $n \in \mathbb{N}$ and $z \in \{0,1,\ldots,y-1\}$.

The entire sequence depends on the “seed” $X_0$.

“Bad” parameters must be avoided:
• $a \neq 0$
• If $b = 0$, then we must choose $X_0 \neq 0$.
• $a \neq 1$ (too predictable)

By definition $U_i \in \{0, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M-1}{M}\}$, i.e. $M =$ number of possible values of $U_i$. Hence, $M$ should be chosen very large.

Since $X_i \in \{0, 1, \ldots, M-1\}$, the sequence $(X_i)_i$ is periodic with period $\leq M$.

MATLAB routine mcg16807: $a = 7^{5}, b = 0, M = 2^{31} - 1$, period $= 2^{31} - 2 \approx 2 \cdot 10^9$. Too small!

Method 2: Fibonacci generator

Choose $k, l, M \in \mathbb{N}$, let $m = \max\{k, l\}$, generate $X_1, \ldots, X_{m-1}$ with Method 1

\[ X_i = (X_{i-k} + X_{i-l}) \mod M, \quad U_i = \frac{X_i}{M} \quad (i = m, m + 1, m + 2, \ldots) \]

MATLAB routine mlfg6331_64: $k = 31, l = 63, M = 2^{64}$, period $\approx 2^{124} > 2 \cdot 10^{37}$

Method 3: Combined multiple recursive generator

Choose $M_1, M_2, M \in \mathbb{N}$ very large, $a, b, c, d \in \mathbb{N}$ large

\[ X_i = (aX_{i-2} - bX_{i-3}) \mod M_1 \]
\[ Y_i = (cY_{i-1} - dY_{i-3}) \mod M_2 \]
\[ Z_i = (X_i - Y_i) \mod (M - 1) \]
\[ U_i = \begin{cases} Z_i/M & \text{if } Z_i \neq 0 \\ (M - 1)/M & \text{if } Z_i = 0 \end{cases} \]

MATLAB routine mrg32k3a, parameters on p. 108 in [GJ10].

Remark: There are many more methods. The default algorithm in MATLAB is called Mersenne Twister mt19937ar.

6.1.2 Normal pseudo-random numbers

Idea: Transform uniform pseudo-random numbers to obtain normal pseudo-random numbers.

Method 1: Inversion

Let $U \sim \mathcal{U}(0, 1)$, i.e. $P(U \leq x) = x$ for all $x \in [0, 1]$.

Let $F : \mathbb{R} \rightarrow [0, 1]$ be a strictly increasing probability distribution. Hence, $F^{-1} : [0, 1] \rightarrow \mathbb{R}$ exists, and if $X := F^{-1}(U)$, then

\[ P(X \leq x) = P(U \leq F(x)) = F(x). \]
Hence, $F$ is the distribution of $X$. Apply this to the normal distribution

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} \, ds.$$ 

Problem: No explicit formula for $F$ or $F^{-1}$. Numerical inversion with Newton’s method is ill-conditioned: If $u \approx 0$ or $u \approx 1$, then small perturbations cause large perturbations of $F^{-1}(u)$.

**Method 2: Box-Muller method**

Let $X \in \mathbb{R}^d$ be a random variable with density $f : \mathbb{R}^d \to \mathbb{R}$, and let $A := \{ x \in \mathbb{R}^d : f(x) > 0 \}$. Let $g : A \to B := g(A) \subset \mathbb{R}^d$ be invertible with continuously differentiable inverse $g^{-1}$. If $Y = g(X)$, then

$$\mathbb{P}(Y \in C) = \mathbb{P}(g(X) \in C) = \mathbb{P}(X \in g^{-1}(C)) = \int_{g^{-1}(C)} f(x) \, dx$$

for all Borel sets $C \subset B$. $J_{g^{-1}}$ denotes the Jacobian of $g^{-1}$. Hence, the function

$$y \mapsto f(g^{-1}(y)) \cdot |\det J_{g^{-1}}(y)|$$

is the density of $Y = g(X)$.

Use this to transform the uniform distribution to the normal distribution. Let $d = 1$, $A = [0, 1]$, $f(x) = 1_A(x)$ and seek $g$ such that for all $y \in B$

$$f(g^{-1}(y)) \cdot |\det J_{g^{-1}}(y)| = \left| \frac{dg^{-1}}{dy}(y) \right| \equiv \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Problem: No explicit formula for $g$. Idea: Transform in $\mathbb{R}^2$ instead of $\mathbb{R}$.

Let $A = (0, 1) \times (0, 1)$, $f(x) = 1_A$ and

$$g(x) = \sqrt{-2 \ln x_1} \begin{pmatrix} \cos(2\pi x_2) \\ \sin(2\pi x_2) \end{pmatrix}, \quad x = (x_1, x_2) \in A.$$ 

The inverse $g^{-1} : B \to (0, 1) \times (0, 1)$ with $B = g(A)$ is given by (exercise)

$$g^{-1}(y) = \begin{pmatrix} \exp \left( -\frac{1}{2}(y_1^2 + y_2^2) \right) \\ H(y_1, y_2) \end{pmatrix}$$
with

\[
H(y_1, y_2) = \begin{cases}
\frac{1}{2\pi} \arctan \left( \frac{y_2}{y_1} \right), & y_1 > 0, \ y_2 > 0, \\
\frac{1}{2\pi} \arctan \left( \frac{y_2}{y_1} \right) + \frac{1}{2}, & y_1 < 0, \\
\frac{1}{2\pi} \arctan \left( \frac{y_2}{y_1} \right) + 1, & y_1 > 0, \ y_2 < 0, \\
\frac{3}{4}, & y_1 = 0, \ y_2 < 0, \\
\frac{1}{4}, & y_1 = 0, \ y_2 > 0, \\
\frac{1}{2}, & y_2 = 0.
\end{cases}
\]

It can be shown (exercise) that

\[
|\det J_{g^{-1}}(y)| = \frac{1}{2\pi} \exp \left( -\frac{1}{2} (y_1^2 + y_2^2) \right)
\]

is the density of the standard normal distribution in \(\mathbb{R}^2\). Hence:

\[
g_1(X), g_2(X) \sim \mathcal{N}(0, 1) \iff X_1, X_2 \sim \mathcal{U}(0, 1).
\]

**Box-Muller algorithm:** Generate uniformly distributed random numbers \(U_1, U_2 \sim \mathcal{U}(0, 1)\) and let

\[
Z_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2) \sim \mathcal{N}(0, 1), \quad Z_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2) \sim \mathcal{N}(0, 1)
\]

G. E. P. Box and M. E. Muller 1958

**Method 3: Polar method**

Goal: Avoid trigonometric functions.

If \(U_i \sim \mathcal{U}(0, 1)\), then \(V_i = 2U_i - 1 \sim \mathcal{U}(-1, 1)\). Reject \((V_1, V_2)\) if \(V := V_1^2 + V_2^2 \geq 1\) or if \(V = 0\).

The accepted samples are uniformly distributed on the unit circle (with density \(f(x) = 1/\pi\)), and it can be shown (exercise) that

\[
\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} V \\ H(V_1, V_2) \end{pmatrix}
\]

is uniformly distributed on \((0, 1) \times (0, 1)\). Hence,

\[
Z_1 = \sqrt{-2 \ln W_1} \cos(2\pi W_2) \sim \mathcal{N}(0, 1), \quad Z_2 = \sqrt{-2 \ln W_1} \sin(2\pi W_2) \sim \mathcal{N}(0, 1)
\]

and by definition

\[
\cos(2\pi W_2) = \frac{V_1}{\sqrt{V}}, \quad \sin(2\pi W_2) = \frac{V_2}{\sqrt{V}}.
\]

**Polar method:** For \(i \in \{1, 2\}\) generate uniform random numbers \(U_i \in (0, 1)\) and let \(V_i = 2U_i - 1\).
• If $V := V_1^2 + V_2^2 \geq 1$: reject and start again.
• Else: Let
  \[
  Z_1 = \frac{V_1}{\sqrt{V}} \sqrt{-2 \ln V} \sim \mathcal{N}(0, 1), \quad Z_2 = \frac{V_2}{\sqrt{V}} \sqrt{-2 \ln V} \sim \mathcal{N}(0, 1).
  \]

G. Marsaglia

The probability that $V < 1$ is $\pi/4$. Hence, about $21.46\%$ of the random tuples $(V_1, V_2)$ are rejected. Nevertheless, the polar method is usually more efficient than the standard Box-Muller method.

### 6.1.3 Correlated normal random vectors

Let $X(\omega) \in \mathbb{R}^d$, $\mu \in \mathbb{R}^d$ and let $\Sigma \in \mathbb{R}^{d \times d}$ be symmetric and positive definite.

Goal: Generate random vectors $X \sim \mathcal{N}(\mu, \Sigma)$, i.e.

\[
\mathbb{P}(X \in B) = \int_B \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \, dx
\]

for all Borel sets $B \subset \mathbb{R}^d$; cf. Definition 2.2.1. The matrix

\[
\rho \in \mathbb{R}^{d \times d} \quad \text{with entries} \quad \rho_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii} \Sigma_{jj}}}
\]

is called the correlation matrix.

Reminder: Every symmetric, positive definite matrix $\Sigma \in \mathbb{R}^{d \times d}$ has a **Cholesky decomposition**

\[
\Sigma = LL^T, \quad L = \begin{pmatrix}
* & 0 & \cdots & \cdots & 0 \\
* & * & 0 & \cdots & : \\
* & \vdots & \ddots & \ddots & : \\
* & \vdots & \ddots & * & \vdots \\
* & \cdots & \cdots & \cdots & *
\end{pmatrix}, \quad L_{ij} = 0 \text{ if } i < j.
\]

**Proof** by induction (exercise).

If $z \in \mathbb{R}^d$ and $x = Lz$, then

\[
z^T z = (L^{-1} x)^T (L^{-1} x) = x^T (LL^T)^{-1} x = x^T \Sigma^{-1} x.
\]

For $A \subset \mathbb{R}^d$ and $B := \{ z = L^{-1} x, x \in A \}$ we have

\[
\int_B \frac{1}{\sqrt{(2\pi)^d}} \exp \left( -\frac{1}{2} z^T z \right) \, dz = \int_A \frac{1}{\sqrt{(2\pi)^d} \det(L)} \left| \det(L) \right| \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right) \, dx
\]

\[
= \int_A \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right) \, dx
\]
because \((\det(L))^2 = \det(L) \cdot \det(L^T) = \det(\Sigma)\). Consequence:

\[
Z \sim \mathcal{N}(0, I) \implies X = LZ \sim \mathcal{N}(0, \Sigma) \implies X + \mu \sim \mathcal{N}(\mu, \Sigma).
\]

### 6.2 Monte Carlo integration and variance reduction

**Example:** European basket call with \(d \in \mathbb{N}\) underlyings modelled by geometric Brownian motion

\[
dS(t) = rSdt + \text{diag}(\sigma)\text{diag}(S)LdW(t), \quad S(t_*) = S_*
\]

with \(\sigma = (\sigma_1, \ldots, \sigma_d), \ r > 0, \ S = (S_1, \ldots, S_d), \ dW = (dW_1, \ldots, dW_d)\) and a lower triangular matrix \(L \in \mathbb{R}^{d \times d}\).

Payoff function (cf. 1.2):

\[
\psi(x) = \left(\sum_{i=1}^{d} c_i x_i - K\right)^+, \quad c_i > 0
\]

As in the scalar case (cf. 3.4), it can be shown that the value of the option is the discounted expected payoff

\[
V(t_*, S_*) = e^{-r(T-t_*)}\mathbb{E}_Q \left(\psi(S_T)\right) = e^{-r(T-t_*)} \int_0^\infty \cdots \int_0^\infty \psi(x)\phi(x, \xi, \beta) \, dx_1 \cdots dx_d
\]

where \(\phi\) is the multivariate log-normal distribution. The parameters \(\xi\) and \(\beta\) depend on \(S_*, T-t_*,\) and on the covariance matrix \(\Sigma = LL^T\). In order to price the option, we thus have to approximate the \(d\)-dimensional integral

\[
\int_0^\infty \cdots \int_0^\infty g(x) \, dx_1 \cdots dx_d
\]

**Approximation by quadrature** as in 5.1:

- **Truncation:** \(0 \leq x_{\min}^{(i)} < x_{\max}^{(i)}\) such that

\[
\int_{x_{\min}^{(d)}}^{x_{\max}^{(d)}} \cdots \int_{x_{\min}^{(1)}}^{x_{\max}^{(1)}} g(x) \, dx_1 \cdots dx_d \approx \int_0^\infty \cdots \int_0^\infty g(x) \, dx_1 \cdots dx_d
\]

- **Discretization:** For every \(i = 1, \ldots, d\) choose large \(N^{(i)} \in \mathbb{N}\), let \(h^{(i)} = (x_{\max}^{(i)} - x_{\min}^{(i)})/N^{(i)}\) and \(x_k^{(i)} = x_{\min}^{(i)} + kh^{(i)}\)
• Approximate by quadrature (here: midpoint rule):

\[
\int_{x^{(1)}_{\min}}^{x^{(1)}_{\max}} \cdots \int_{x^{(d)}_{\min}}^{x^{(d)}_{\max}} g(x) \, dx_1 \cdots dx_d
\approx h^{(1)} \cdot \ldots \cdot h^{(d)} \sum_{k_1=1}^{N^{(1)}} \cdots \sum_{k_d=1}^{N^{(d)}} g\left(x^{(1)}_{k_1} + \frac{1}{2} h^{(1)} , \ldots , x^{(d)}_{k_d} + \frac{1}{2} h^{(d)}\right)
\]

**Problem:** Need function evaluations at \(N^{(1)} \cdot \ldots \cdot N^{(d)}\) points, e.g. \(N^d\) evaluations if \(N^{(i)} = N\) for all \(i\). Exponential growth for \(d \to \infty\), “curse of dimension”. Very expensive or impossible for \(d \gg 1\).

**Solutions?** Sparse grids (→ summer term) or (Quasi-)Monte Carlo integration.

### Monte Carlo integration

Consider bounded domain \(D \subset \mathbb{R}^d\), function \(f : D \to \mathbb{R}\), density \(\phi : D \to \mathbb{R}\). As in 5.1 approximate

\[
\mathbb{E}(f) := \int_D f(x)\phi(x) \, dx \approx \frac{1}{m} \sum_{j=1}^{m} f(X_j)
\]

(6.1)

where \(X_j \in D\) are random vectors with

\[
\mathbb{P}(X_j \in A) = \int_A \phi(x) \, dx \quad \text{for all measurable } A \subset D.
\]

**Question:** How accurate is this approximation?

For simplicity consider only the case \(d = 1\). If \(d > 1\), then the following result can be applied to each entry of the random vector.

**Lemma 6.2.1 (Chebyshev’s inequality)** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \(\delta > 0\). If \(Z : \Omega \to \mathbb{R}\) is square integrable, i.e. if \(\int_{\Omega} |Z|^2 \, d\mathbb{P}\) exists, then

\[
\mathbb{P}(\{|Z - \mathbb{E}(Z)| \geq \delta\}) \leq \frac{\mathbb{V}(Z)}{\delta^2}
\]

where \(\mathbb{V}(Z) = \mathbb{E}(|Z - \mathbb{E}(Z)|^2)\) is the variance of \(Z\).

**Proof.** Define

\[
\chi_\delta(\omega) = \begin{cases} 
1 & \text{if } |Z(\omega) - \mathbb{E}(Z)| \geq \delta \\
0 & \text{else}.
\end{cases}
\]
Then \( \chi_\delta(\omega) \leq \frac{|Z(\omega) - \mathbb{E}(Z)|}{\sigma} \) for all \( \omega \in \Omega \) by construction, and hence

\[
\mathbb{P}\left( |Z - \mathbb{E}(Z)| \geq \delta \right) = \int_\Omega \chi_\delta(\omega) \, d\mathbb{P}(\omega) = \int_\Omega \chi_\delta^2(\omega) \, d\mathbb{P}(\omega) \leq \frac{1}{\delta^2} \int_\Omega |Z(\omega) - \mathbb{E}(Z)|^2 \, d\mathbb{P}(\omega) = \frac{\mathbb{V}(Z)}{\delta^2}.
\]

Now let \( Y_m := \frac{1}{m} \sum_{j=1}^m f(X_j) \approx \mathbb{E}(f) \) and

\[
\mathbb{E}(f(X_j)) = \mathbb{E}(f), \quad \mathbb{V}(f(X_j)) = \sigma^2
\]

for all \( j = 1, \ldots, m \) and some \( \sigma > 0 \). Consequence:

\[
\mathbb{E}(Y_m) = \mathbb{E}(f) \quad \mathbb{V}(Y_m) = \frac{\mathbb{V}(f(X_j))}{m} = \frac{\sigma^2}{m}.
\]

Applying Lemma 6.2.1 to \( Y_m \) yields for all \( \delta > 0 \)

\[
\mathbb{P}\left( |Y_m - \mathbb{E}(f)| \geq \delta \right) \leq \frac{\mathbb{V}(Y_m)}{\delta^2} = \frac{\sigma^2}{\delta^2 m}.
\]

Now choose \( \varepsilon > 0 \) and let \( \delta := \frac{\sigma}{\sqrt{\varepsilon m}} \):

\[
\mathbb{P}\left( |Y_m - \mathbb{E}(f)| \geq \frac{\sigma}{\sqrt{\varepsilon m}} \right) \leq \varepsilon
\]

or equivalently

\[
\mathbb{P}\left( |Y_m - \mathbb{E}(f)| < \frac{\sigma}{\sqrt{\varepsilon m}} \right) > 1 - \varepsilon.
\]

Interpretation:

Good approximation with high probability \( \iff \varepsilon \) small, \( \frac{\sigma}{\sqrt{\varepsilon m}} \) small.

Slow convergence: For fixed \( \varepsilon \) and \( \sigma \), reducing the error by a factor of 10 comes at the cost of increasing the number of samples by a factor of 100.

**Variance reduction**

Idea: Try to decrease \( \sigma \) to improve the accuracy.
Method 1: Decomposition

Let \( g : \mathcal{D} \rightarrow \mathbb{R} \) be a function such that

\[
\mathbb{E}(g) = \int_{\mathcal{D}} g(x) \phi(x) \, dx \approx \int_{\mathcal{D}} f(x) \phi(x) \, dx = \mathbb{E}(f)
\]

and such that \( \mathbb{E}(g) \) can be computed analytically. Let

\[
Y_m = \frac{1}{m} \sum_{j=1}^{m} f(X_j) \approx \mathbb{E}(f) \quad \text{(as before)}
\]

\[
Y^*_m = \frac{1}{m} \sum_{j=1}^{m} g(X_j) \approx \mathbb{E}(g)
\]

\[
Z_m(s) = Y_m + s(\mathbb{E}(g) - Y^*_m)
\]

for \( s \in \mathbb{R} \). Then it follows from \( \mathbb{E}(Y_m) = \mathbb{E}(f) \) and \( \mathbb{E}(Y^*_m) = \mathbb{E}(g) \) that

\[
\mathbb{E}(Z_m(s)) = \mathbb{E}(Y_m) + s(\mathbb{E}(g) - \mathbb{E}(Y^*_m)) = \mathbb{E}(f)
\]

for all \( s \in \mathbb{R} \). Moreover,

\[
\mathbb{V}(Z_m(s)) = \mathbb{E}\left( \left( Z_m(s) - \mathbb{E}(f) \right)^2 \right)
\]

\[
= \mathbb{E}\left( \left( Y_m - \mathbb{E}(f) - s(Y^*_m - \mathbb{E}(g)) \right)^2 \right)
\]

\[
= \mathbb{V}(Y_m) - 2s \text{Cov}(Y_m, Y^*_m) + s^2 \mathbb{V}(Y^*_m)
\]

where \( \text{Cov}(A, B) \) is the covariance of two random variables \( A \) and \( B \), i.e.

\[
\text{Cov}(A, B) = \mathbb{E}\left( (A - \mathbb{E}(A))(B - \mathbb{E}(B)) \right).
\]

Hence, \( \mathbb{V}(Z_m(s)) \) is minimal for

\[
s = s_{\text{min}} := \frac{\text{Cov}(Y_m, Y^*_m)}{\mathbb{V}(Y^*_m)}
\]

and takes the value

\[
\mathbb{V}(Z_m(s_{\text{min}})) = \mathbb{V}(Y_m) - \frac{\text{Cov}(Y_m, Y^*_m)^2}{\mathbb{V}(Y^*_m)} = (1 - \rho^2)\mathbb{V}(Y_m)
\]

with correlation coefficient

\[
\rho = \frac{\text{Cov}(Y_m, Y^*_m)}{\sqrt{\mathbb{V}(Y_m)\mathbb{V}(Y^*_m)}} \in [-1, 1].
\]

Hence, the estimator \( Z_m(s_{\text{min}}) \) has a much smaller variance than \( Y_m \) if \( \rho \approx \pm 1 \). The optimal value \( s_{\text{min}} \) is usually not known, but if \( g \approx f \), then \( s = 1 \) is often a good choice.
Method 2: Antithetic variates

Assumption: \( Y_m \) is generated with normal random variables \( X_j \sim \mathcal{N}(0, 1) \).

Since \( -X_j \sim \mathcal{N}(0, 1) \), too, we define \( Y_m^{-} := \frac{1}{m} \sum_{j=1}^{m} f(-X_j) \approx \mathbb{E}(f) \) and put \( \hat{Y}_m = \frac{1}{2}(Y_m + Y_m^{-}) \).

Applying

\[
0 \leq \mathbb{V}(A \pm B) = \mathbb{V}(A) + \mathbb{V}(B) \pm 2\mathbb{Cov}(A, B)
\]

with “+” we obtain

\[
\mathbb{V}(\hat{Y}_m) = \frac{1}{4} \mathbb{V}(Y_m + Y_m^{-}) = \frac{1}{4} \left( \mathbb{V}(Y_m) + \mathbb{V}(Y_m^{-}) + 2\mathbb{Cov}(Y_m, Y_m^{-}) \right)
\]

\[
= \frac{1}{2} \left( \mathbb{V}(Y_m) + \mathbb{Cov}(Y_m, Y_m^{-}) \right).
\]

(6.3)

If \( \mathbb{Cov}(Y_m, Y_m^{-}) > 0 \), then (6.2) with “-” yields

\[
\mathbb{Cov}(Y_m, Y_m^{-}) \leq \frac{1}{2} \left( \mathbb{V}(Y_m) + \mathbb{V}(Y_m^{-}) \right) = \mathbb{V}(Y_m)
\]

and it follows from (6.3) that

\[
\mathbb{V}(\hat{Y}_m) \leq \mathbb{V}(Y_m) \quad \text{(\( \implies \) at least not worse)}.
\]

If \( \mathbb{Cov}(Y_m, Y_m^{-}) \leq 0 \), then (6.3) yields

\[
\mathbb{V}(\hat{Y}_m) \leq \frac{1}{2} \mathbb{V}(Y_m) \quad \text{(\( \implies \) smaller variance)}.
\]

Variance reduction by antithetic variates for SDEs

For SDE-based Monte Carlo methods (cf. section 5.5), the mean-square-error depends on \( \mathbb{V}(\psi(S_N)) \). This can be reduced by approximating the SDE

\[
dS(t) = f(t, S(t))dt + g(t, S(t))dW(t), \quad t \in [0, T].
\]

with Euler-Maruyama and antithetic variates:

\[
S_{n+1}^+ = S_n^+ + \tau f(t_n, S_n^+) + g(t_n, S_n^+) \Delta W_n
\]

\[
S_{n+1}^- = S_n^- + \tau f(t_n, S_n^-) - g(t_n, S_n^-) \Delta W_n, \quad n = 0, \ldots, N - 1,
\]

and then use the values \( S_n = \frac{1}{2}(S_n^+ + S_n^-) \).
6.3 Quasi-Monte Carlo methods

Let $\mathcal{D} = [0,1]^d$ be the $d$-dimensional unit cube and consider the special case $\phi(x) \equiv 1$ in (6.1):

$$
E(f) := \int_{\mathcal{D}} f(x) \, dx \approx \frac{1}{m} \sum_{j=1}^{m} f(X_j)
$$

(Monte Carlo integration with uniformly distributed random vectors $X_j \in \mathcal{D}$)

**Problem:** Tuples of uniform random numbers are usually not homogeneously distributed in space.
Quasi-Monte Carlo methods use the same formula, but replace the random vectors $X_j$ by deterministic low-discrepancy point sequences.

**Definition 6.3.1 (Discrepancy)**

1. Let $\mathcal{R}$ be the set of all axially parallel $d$-dimensional rectangles $R \subset \mathcal{D}$. The **discrepancy** of the points $x_1, \ldots, x_m \in \mathcal{D} \subset \mathbb{R}^d$ is

$$
D_m := \sup_{R \in \mathcal{R}} \left| \frac{\# \text{ of } x_i \text{ in } R}{m} - \frac{\text{vol}(R)}{m} \right|
$$

where $\text{vol}(R) = \int_R 1 \, dx$ denotes the volume of $R$.

2. The **star discrepancy** $D^*_m$ is defined as $D_m$, but the supremum is only taken over those $R$ for which $(0, \ldots, 0)$ is one of the corners.

3. A sequence $(x_k)_{k \in \mathbb{N}}$ of points $x_k \in \mathcal{D}$ is called **low-discrepancy sequence** if

$$
D_m = \mathcal{O}\left( \frac{\log m^d}{m} \right) \quad \text{for } m \to \infty.
$$

In this case, the $x_k$ are called **quasi-random vectors**.

**Properties (without proofs):**

- $D^*_m \leq D_m \leq 2^d D^*_m$ (exercise)
- The Koksma-Hlawka-Theorem provides the deterministic error bound

$$
\left| E(f) - \frac{1}{m} \sum_{j=1}^{m} f(x_j) \right| \leq TV(f) \cdot D^*_m
$$

where $TV(f)$ is the total variation of $f$; cf. (2.2). Numerical tests show that this bound is often too pessimistic.
It can be shown that

$$\mathbb{E}(D_m) = O\left(\sqrt{\frac{\log \log m}{m}}\right)$$

for \textit{randomly} chosen sequences.

Consequence: The error of a quasi-Monte Carlo method decays like \((\log m)^d/m\), whereas the error of standard Monte Carlo is expected to decay like \(\sqrt{\log \log m}/\sqrt{m}\). If we use the midpoint rule with mesh-width \(h = 1/N\) in each spatial dimension, then the total number of points is \(m = N^d\), and the error scales like \(h^2 = m^{-2/d}\).

Question: How can low-discrepancy sequences be generated?

**Example:** For \(d = 1\) the sequence with

$$x_j = \frac{2j - 1}{2m}, \quad j = 1, \ldots, m$$

has \(D_m^* = 1/(2m)\). This value is optimal. But: This sequence can only be used if \(m\) is known \textit{a priori}, and if \(m\) is changed, then all values change.

**Van der Corput sequence:**

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$$

Algorithm: Represent the index \(j \in \mathbb{N}\) as a binary number

$$j = \sum_{k=0}^{L} d_k 2^k = (d_L d_{L-1} \ldots d_1 d_0)_2, \quad d_k \in \{0, 1\}$$
and define

\[ \eta_2(j) = \sum_{k=0}^{L} d_k 2^{-k-1} = (d_0 d_1 \ldots d_L)_2, \quad \text{for } d_k \in \{0, 1\}. \]

Interpretation: Reverse binary digits and put the radix point in front of the sequence. Example: \( j = 6 \) yields \( d_2 = d_1 = 1, \ d_0 = 0 \) and hence \( \eta_2(6) = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \).

**Generalization:** For an arbitrary base \( b \in \mathbb{N} \) define the radical-inverse function

\[ \eta_b(j) = \sum_k d_k b^{-k-1} \]

where \( d_k \in \{0, 1, \ldots, b - 1\} \) are the coefficients from the representation \( j = \sum_k d_k b^k \).

The **Halton sequence** generates quasi-random vectors in the hypercube \( D = [0, 1]^d \) by letting

\[ x_j = (\eta_{p_1}(j), \ldots, \eta_{p_d}(j)) \]

where \( p_1, \ldots, p_d \) are prime numbers with \( p_i \neq p_j \) for \( i \neq j \). It was shown that this is indeed a low-discrepancy sequence.

Other possibility: Sobol sequence
Chapter 7

Finite-difference methods for parabolic differential equations

7.1 Motivation and model problem

Reminder (cf. 3.2): The value of an European option is the solution of the Black-Scholes equation

\[ \frac{\partial V(t, S)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V(t, S)}{\partial S^2} + r S \frac{\partial V(t, S)}{\partial S} - r V(t, S) = 0 \]

with terminal condition

\[ V(T, S) = \psi(S) \quad \text{(payoff function)}. \]

Notation: \( T > 0 \) maturity, \( r > 0 \) interest rate, \( \sigma \in \mathbb{R} \) volatility, \( S \) price of the underlying.

More complicated market models (e.g. with volatility \( \sigma = \sigma(t, S) \)) lead to similar PDEs for which no solution formulas are available; cf. 3.5.

Question: Numerical methods?

Basic types of PDEs

- **Elliptic PDEs**: Poisson equation

  \[ -\Delta u(x) = f(x), \quad f(x) \text{ given}, \quad \Delta u = \sum_{k=1}^{d} \frac{\partial^2 u(x)}{\partial x^2_k} \text{ Laplace operator} \]

- **Parabolic PDEs**: Heat equation

  \[ \partial_t u(t, x) = \Delta u(t, x) \]
• Hyperbolic PDEs: Wave equation

\[ \partial_t^2 u(t, x) = \Delta u(t, x) \]

The Black-Scholes equation is a parabolic PDE and can be transformed to the heat equation; cf. 3.3. Therefore, we will consider the following

**Model problem:** Heat equation on an interval with Dirichlet boundary conditions.

\[ \begin{align*}
    \partial_t u(t, x) &= \partial_x^2 u(t, x) & t \in (0, t_{\text{end}}], x \in (a, b) & \text{PDE} (7.1a) \\
    u(t, a) &= u_a(t), \quad u(t, b) &= u_b(t) & t \in [0, t_{\text{end}}] & \text{boundary conditions} (7.1b) \\
    u(0, x) &= u_0(x) & x \in [a, b] & \text{initial condition} (7.1c)
\end{align*} \]

The parameters \(a, b, t_{\text{end}},\) the boundary values \(u_a(t)\) and \(u_b(t)\) and the initial data \(u_0(x)\) are given.

**Notation:** We say that \(f \in C^j([a, b])\) if and only if \(x \mapsto f(x)\) is \(j\) times continuously differentiable on \((a, b)\), and all derivatives can be extended to \([a, b]\). \(C^0_0([a, b])\) is the subspace of all \(f \in C^j([a, b])\) with \(f(a) = f(b) = 0\). Moreover, let

\[ \|f\|_\infty = \max_{x \in [a, b]} |f(x)| \]

denote the maximum norm on \([a, b]\).

### 7.2 Space discretization with finite differences

Choose \(1 < m \in \mathbb{N}\), let \(h = (b - a)/m\) and \(x_k = a + k \cdot h\) for \(k = 0, \ldots, m\), and define

\[ \bar{u}(t) = \left( u(t, x_1), \ldots, u(t, x_{m-1}) \right)^T \in \mathbb{R}^{m-1}. \]

The entries of \(\bar{u}(t)\) are the values of the exact solution at the inner grid points at time \(t\).

**Goal:** Find \(w_h : [0, t_{\text{end}}] \to \mathbb{R}^{m-1}\) such that \(w_h(t) \approx \bar{u}(t)\) for all \(t \in [0, t_{\text{end}}]\).

Approximate spatial derivatives by difference quotients.

**Lemma 7.2.1 (difference quotients)** For arbitrary \(k \in \{1, \ldots, m - 1\}\), the (first or second) derivative of a function \(f : [a, b] \to \mathbb{R}\) in \(x_k\) can be approximated as follows:

- If \(f \in C^2([a, b])\), then

\[ \max_{k=1,\ldots,m-1} \left| \frac{f'(x_k) - f'(x_{k+1})}{h} \right| \leq C h \|f''\|_\infty \]

\[ \max_{k=1,\ldots,m-1} \left| \frac{f'(x_k) - f'(x_{k-1})}{h} \right| \leq C h \|f''\|_\infty \]
• If \( f \in C^3([a,b]) \), then
\[
\max_{k=1,...,m-1} \left| f'(x_k) - \frac{f(x_{k+1}) - f(x_{k-1})}{2h} \right| \leq Ch^2 \left\| \frac{d^3 f}{dx^3} \right\|_{\infty}
\]

• If \( f \in C^4([a,b]) \), then
\[
\max_{k=1,...,m-1} \left| f''(x_k) - \frac{f(x_{k+1}) - 2f(x_k) + f(x_{k-1})}{h^2} \right| \leq Ch^2 \left\| \frac{d^4 f}{dx^4} \right\|_{\infty} \tag{7.2}
\]

The constant \( C \) is independent of \( h \) and can have different values in each case.

**Proof:** Use Taylor’s theorem (exercise).

Applying (7.2) to the heat equation yields
\[
\partial_t u(t,x_k) = \partial_x^2 u(t,x_k) \approx \frac{u(t,x_{k+1}) - 2u(t,x_k) + u(t,x_{k-1})}{h^2} \tag{7.3}
\]
for all \( k = 1, \ldots, m-1 \) and \( t \in (0,t_{\text{end}}] \). The boundary values \( u(t,x_0) = u_a(t) \) and \( u(t,x_m) = u_b(t) \) are known from (7.1b).

Reformulation in matrix-vector notation: Define the vector
\[
g_h(t) = \frac{1}{h^2} \left( u_a(t), 0, 0, \ldots, 0, 0, u_b(t) \right)^T \in \mathbb{R}^{m-1}
\]
and the matrix
\[
A_h = \frac{1}{h^2} \begin{pmatrix}
-2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
1 & -2 & 1 & 0 & \vdots & & \\
0 & 1 & -2 & 1 & 0 & \vdots & \\
& \ddots & \ddots & \ddots & \ddots & \vdots & \\
& & 0 & 1 & -2 & 1 & 0 \\
& & & \ddots & \ddots & \ddots & \ddots \\
& & & & 0 & 1 & -2 & 1 \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & -2 & 1
\end{pmatrix} \in \mathbb{R}^{(m-1) \times (m-1)}. \tag{7.4}
\]

Then (7.3) reads (check!)
\[
\tilde{u}'(t) \approx A_h \tilde{u}(t) + g_h(t).
\]

The approximation \( w_h(t) \approx \tilde{u}(t) \) is now defined as the solution of the initial value problem
\[
\begin{align*}
w'_h(t) &= A_h w_h(t) + g_h(t) \tag{7.5a} \\
w_h(0) &= \tilde{u}(0). \tag{7.5b}
\end{align*}
\]

Hence, the space discretization turns the PDE into an ordinary differential equation (ODE).
Properties of the discretization matrix

Lemma 7.2.2 For \( k = 1, \ldots, m - 1 \), the vectors

\[
\nu_k := \left( \sin \left( \frac{k\pi}{m} \right), \sin \left( \frac{2k\pi}{m} \right), \ldots, \sin \left( \frac{(m-1)k\pi}{m} \right) \right)^T \in \mathbb{R}^{m-1}
\]

are eigenvectors of the matrix \( A_h \), and the corresponding eigenvalues are

\[
\lambda_{h,k} := \frac{2}{h^2} \left( \cos \left( \frac{kh\pi}{b-a} \right) - 1 \right).
\]

Proof: Exercise.

Remark. The eigenfunctions of the operator

\[
\frac{\partial^2}{\partial x^2} : C^2_c([a,b]) \rightarrow C([a,b])
\]

are \( \phi_k(x) := \sin \left( \pi k \frac{x-a}{b-a} \right) \) for \( k \in \mathbb{N} \), and the eigenvalues are \( \sigma_k := -\left( \frac{\pi k}{b-a} \right)^2 \). The entries of \( \nu_k \) are simply the values of \( \phi_k \) on the grid, i.e.

\[
\nu_k = (\phi_k(x_1), \ldots, \phi_k(x_{m-1})).
\]

For the eigenvalues we have for \( h \to 0 \)

\[
|\lambda_{h,k} - \sigma_k| = \frac{2}{h^2} \left( \cos \left( \frac{kh\pi}{b-a} \right) - 1 \right) + \frac{1}{2} \left( \frac{\pi hk}{b-a} \right)^2 = \mathcal{O} \left( h^2 k^4 \right).
\]

Notation. Henceforth, \( |v| := \sqrt{v^T v} \) denotes the Euclidean norm of a vector \( v \) or the induced matrix norm, respectively. (If \( v \) is a scalar, then the norm coincides with the absolute value.) Let \( I \) be the identity matrix. The exponential of a matrix \( M \) is defined by

\[
e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}
\]

with convergence in the matrix norm.

Lemma 7.2.3 For any \( s \geq 0 \), \( 1 < m \in \mathbb{N} \) and \( h = (b-a)/m \) we have

\[
|e^{sA_h}| \leq 1, \quad |(I - sA_h)^{-1}| \leq 1.
\]

Proof. Since \( A_h \) is symmetric, there is an orthogonal matrix \( Q_h \) such that \( A_h = Q_h \Lambda_h Q_h^T \) with

\[
\Lambda_h = \text{diag}(\lambda_{h,1}, \ldots, \lambda_{h,m-1}), \quad \lambda_{h,k} = \frac{2}{h^2} \left( \cos \left( \frac{kh\pi}{b-a} \right) - 1 \right) < 0.
\]
Hence,

\[ A_h^2 = Q_h \Lambda h Q_h^T Q_h \Lambda h Q_h^T = Q_h \Lambda h^2 Q_h^T, \quad A_h^k = Q_h \Lambda h^k Q_h^T \]

and as a consequence

\[ e^{s A_h} = \sum_{k=0}^{\infty} Q_h \frac{(s \Lambda h)^k}{k!} Q_h^T = Q_h e^{s \Lambda h} Q_h^T. \]

Since \( e^{s \Lambda h} \) is a diagonal matrix and \( s \lambda_{h,k} \leq 0 \), we have

\[ |e^{s \Lambda h}| = \sup_{k=1,\ldots,m-1} |e^{s \lambda_{h,k}}| \leq 1. \]

This yields the bound

\[ |e^{s A_h}| \leq |Q_h| \cdot |e^{s \Lambda h}| \cdot |Q_h^T| \leq 1 \]

because \( |Q_h| = 1 \). In a similar way, it can be checked that

\[ (I - s A_h)^{-1} = Q_h (I - s \Lambda h)^{-1} Q_h^T \]

which yields the bound

\[ |(I - s A_h)^{-1}| \leq |Q_h| \cdot |(I - s \Lambda h)^{-1}| \cdot |Q_h^T| = \max_{k=1,\ldots,m-1} \frac{1}{1 - s \lambda_{h,k}} \leq 1 \]

since \( s \lambda_{h,k} \leq 0 \).

\[ \square \]

**Equivalent reformulation**

Let \( u \) be the solution of (7.1):

\[
\begin{align*}
\partial_t u(t, x) &= \partial^2_x u(t, x) & t \in (0, t_{\text{end}}], \ x \in (a, b) & (7.1a) \\
u(t, a) &= u_a(t), \ u(t, b) = u_b(t) & t \in [0, t_{\text{end}}] & (7.1b) \\
u(0, x) &= u_0(x) & x \in [a, b] & (7.1c)
\end{align*}
\]

Let \( \tilde{u} \) be the linear interpolation of the (known) boundary data, i.e.

\[
\begin{align*}
\tilde{u}(t, x) &= u_a(t) + \frac{u_b(t) - u_a(t)}{b - a} (x - a), \\
\tilde{u}(t, a) &= u_a(t), \quad \tilde{u}(t, b) = u_b(t), \quad \partial^2_x \tilde{u}(t, x) = 0.
\end{align*}
\]
Then, \( \hat{u} = u - \tilde{u} \) is the solution of

\[
\partial_t \hat{u}(t, x) = \partial_x^2 \hat{u}(t, x) + f(t, x) \quad t \in (0, t_{\text{end}}], x \in (a, b) \tag{7.6a}
\]

\[
\hat{u}(t, a) = \hat{u}(t, b) = 0 \quad t \in [0, t_{\text{end}}] \tag{7.6b}
\]

\[
\hat{u}(0, x) = \hat{u}_0(x) \quad x \in [a, b] \tag{7.6c}
\]

with \( f(t, x) = -\partial_x \tilde{u}(t, x) \) and \( \hat{u}_0(x) = u_0(x) - \tilde{u}(0, x) \). Miniproof:

\[
\partial_t \hat{u}(t, x) = \partial_t u(t, x) - \partial_t \tilde{u}(t, x) = \partial_x^2 u(t, x) + f(t, x) = \partial_x^2 \hat{u}(t, x) + \partial_x^2 \tilde{u}(t, x) + f(t, x).
\]

Hence, the homogenous PDE (7.1a) with inhomogenous boundary conditions (7.1b) is equivalent to a inhomogenous PDE with homogenous boundary conditions. Do the same for the spatially discretized problem. Let

\[
\tilde{w}(t) = \left( \tilde{u}(t, x_1), \ldots, \tilde{u}(t, x_m) \right)^T \in \mathbb{R}^m
\]

contain the values of \( \tilde{u}(t, \cdot) \) at the grid points, and let \( w_h \) be again the solution of (7.5)

\[
w'_h(t) = A_h w_h(t) + g_h(t) \tag{7.5a}
\]

\[
w_h(0) = \tilde{u}(0). \tag{7.5b}
\]

Then, \( \hat{w}_h = w_h - \tilde{w} \) is the solution of

\[
\hat{w}'_h(t) = A_h \hat{w}_h(t) + \tilde{f}(t) \tag{7.7a}
\]

\[
\hat{w}_h(0) = w_h(0) - \tilde{w}(0) \tag{7.7b}
\]

with \( \tilde{f}(t) = \left( f(t, x_1), \ldots, f(t, x_m) \right)^T \in \mathbb{R}^m \) (exercise).

From now on, we will consider (7.6) instead of (7.1) and (7.7) instead of (7.5). In order to keep the notation simple, however, we will omit the \(^\wedge\) everywhere. Hence, we consider the problem

\[
\partial_t u(t, x) = \partial_x^2 u(t, x) + f(t, x) \quad t \in (0, t_{\text{end}}], x \in (a, b) \tag{7.8a}
\]

\[
u(t, a) = v(t, b) = 0 \quad t \in [0, t_{\text{end}}] \tag{7.8b}
\]

\[
u(0, x) = u_0(x) \quad x \in [a, b]. \tag{7.8c}
\]

A function \( u \) is called a (classical) solution of (7.8) if \( \partial_t u, \partial_x u, \partial_x^2 u \) exist and are continuous on \((0, t_{\text{end}}) \times (a, b)\), and if \( u \) is continuous on \([0, t_{\text{end}}] \times [a, b]\).

Let \( \bar{u}(t) \) be the vector containing the values of \( u(t, \cdot) \) at the grid points, i.e.

\[
\bar{u}(t) = \left( u(t, x_1), \ldots, u(t, x_m) \right)^T \in \mathbb{R}^m.
\]

The spatial semidiscretization of (7.8) reads

\[
w'_h(t) = A_h w_h(t) + \tilde{f}(t) \tag{7.9a}
\]

\[
w_h(0) = \bar{u}(0). \tag{7.9b}
\]
Convergence of the space discretization

Next goal: Prove an error bound for the approximation \( w_h(t) \approx \bar{u}(t) \). In which norm? In order to study convergence, we consider the limit \( m \to \infty \), but since \( w_h(t), \bar{u}(t) \in \mathbb{R}^{m-1} \), the length of the error vector \( w_h(t) - \bar{u}(t) \) increases. The disadvantage of the Euclidean norm is the fact that the norm of a constant vector depends on its length: If \( c > 0 \) and \( (c, \ldots, c)^T \in \mathbb{R}^{m-1} \), then

\[
| (c, \ldots, c)^T | = \left( \sum_{i=1}^{m-1} c^2 \right)^{1/2} = c \sqrt{m-1}.
\]

Therefore, it is advantageous to consider a scaled norm:

**Definition 7.2.4 (scaled norm)** Let \( 1 < m \in \mathbb{N} \) and \( h = (b-a)/m \). The scaled norm \( \| \cdot \|_h \) on \( \mathbb{R}^{m-1} \) is defined by

\[
\| v \|_h = \sqrt{h} | v | \quad v \in \mathbb{R}^{m-1}.
\]

The induced matrix norm of a matrix \( M \in \mathbb{R}^{(m-1) \times (m-1)} \) is again denoted by \( |M|_h \). Note that

\[
|M|_h = \sup_{v \neq 0} \frac{|Mv|_h}{\|v\|_h} = \sup_{v \neq 0} \frac{\sqrt{h} |Mv|}{\sqrt{h} |v|} = |M|.
\]

**Motivation:** If \( \varphi \in C([a,b]) \) is continuous with \( \varphi(a) = \varphi(b) = 0 \) and \( \bar{\varphi}_k = \varphi(x_k) \) for \( k = 1, \ldots, m-1 \), then

\[
\| \varphi \|_{L^2([a,b])} = \left( \int_a^b |\varphi(x)|^2 \, dx \right)^{1/2} \approx \left( h \sum_{k=1}^{m-1} |\varphi(x_k)|^2 \right)^{1/2} = |\bar{\varphi}|_h
\]

for sufficiently large \( m \).

**Theorem 7.2.5 (error of the space discretization)** Let \( u = u(t,x) \) be the solution of the model problem (7.8), and let \( w_h(t) \) be the solution of (7.9). Assume that \( u(t,\cdot) \in C^4_0([a,b]) \) for every \( t \in [0, t_{\text{end}}] \) and that

\[
\sup_{s \in [0,t_{\text{end}}]} \| \partial_x^4 u(s, \cdot) \|_{\infty} < \infty.
\]

Then, there is a constant \( C \) such that

\[
| \bar{u}(t) - w_h(t) |_h \leq C t h^2 \sup_{s \in [0,t_{\text{end}}]} \| \partial_x^4 u(s, \cdot) \|_{\infty}
\]

for all \( t \in [0, t_{\text{end}}] \).
Proof. By definition of \( \bar{u} \) we have for \( k = 1, \ldots, m - 1 \)
\[
\bar{u}'_k(t) = \partial_t u(t, x_k) = \partial^2_x u(t, x_k) + f(t, x_k)
\]
\[
= \frac{u(t, x_{k+1}) - 2u(t, x_k) + u(t, x_{k-1})}{h^2} + r_{h,k}(t) + f(t, x_k)
\]
with remainder term \( r_{h,k}(t) \) bounded by
\[
|r_{h,k}(t)| \leq Ch^2 \| \partial^4_x u(t, \cdot) \|_\infty
\]
according to Lemma 7.2.1. Setting \( r_h := (r_{h,1}, \ldots, r_{h,m-1})^T \), this is equivalent to
\[
\bar{u}'(t) = A_h \bar{u}(t) + r_h(t) + \bar{f}(t).
\]
Comparing with (7.9) shows that the error solves the ODE
\[
\frac{d}{dt}(\bar{u}(t) - w_h(t)) = A_h (\bar{u}(t) - w_h(t)) + r_h(t).
\]
with initial value \( \bar{u}(0) - w_h(0) = 0 \). The solution is given by the variation-of-constants formula
\[
(\bar{u}(t) - w_h(t)) = e^{tA_h} (\bar{u}(0) - w_h(0)) + \int_0^t e^{(t-s)A_h} r_h(s) \, ds.
\]
Hence, Lemma 7.2.2 and (7.11) yield
\[
|\bar{u}(t) - w_h(t)|_h \leq \int_0^t \left| e^{(t-s)A_h} \right| \| r_h(s) \|_h \, ds,
\]
and with
\[
|r_h(s)|_h = \sqrt{h} |r_h(s)| \leq \sqrt{h} \sqrt{m-1} \max_{i=1,\ldots,m-1} |r_{h,i}(s)| \leq \sqrt{b - a \cdot Ch^2} \| \partial^4_x u(s, \cdot) \|_\infty
\]
it follows that
\[
|\bar{u}(t) - w_h(t)|_h \leq Ch^2 \sqrt{b - a} \int_0^t \| \partial^4_x u(s, \cdot) \|_\infty \, ds
\]
\[
\leq Ch^2 \sqrt{b - a} \max_{s \in [0, t_{\text{end}}]} \| \partial^4_x u(s, \cdot) \|_\infty.
\]

Remark. The regularity assumptions are very strong and exclude payoff functions as initial data.
7.3 Time discretization

Space discretization turns the heat equation (7.8a) into the ODE

\[ w'_h(t) = A_h w_h(t) + \bar{f}(t) \]  

with initial value \( w_h(0) = \bar{u}(0) \). Choose a step-size \( \tau > 0 \), let \( t_n = n\tau \) and compute approximations \( w^n_h \approx w_h(t_n) \) with a Runge-Kutta method (see Appendix C) and initial data \( w^0_h \approx w_h(0) = \bar{u}(0) \). Examples:

- Explicit Euler method (order 1)
  \[ w^{n+1}_h = w^n_h + \tau A_h w^n_h + \tau \bar{f}(t_n) \]

- Implicit Euler method (order 1)
  \[ w^{n+1}_h = w^n_h + \tau A_h w^{n+1}_h + \tau \bar{f}(t_{n+1}) \] 
  \[ \iff (I - \tau A_h) w^{n+1}_h = w^n_h + \tau \bar{f}(t_{n+1}) \]

- Trapezoidal rule (order 2)
  \[ w^{n+1}_h = w^n_h + \frac{\tau}{2} A_h (w^{n+1}_h + w^n_h) + \frac{\tau}{2} (\bar{f}(t_{n+1}) + \bar{f}(t_n)) \] 
  \[ \iff (I - \frac{\tau}{2} A_h) w^{n+1}_h = (I + \frac{\tau}{2} A_h) w^n_h + \frac{\tau}{2} (\bar{f}(t_{n+1}) + \bar{f}(t_n)) \]

A-stability

For simplicity, assume that \( \bar{f}(t) = 0 \). As in the proof of Lemma 7.2.3 we consider the eigendecomposition

\[ A_h = Q_h \Lambda_h Q_h^T, \quad \Lambda_h = \text{diag}(\lambda_{h,1}, \ldots, \lambda_{h,m-1}), \quad \lambda_{h,k} < 0 \]  

with \( |Q_h| = |Q_h| = 1 \). Then, the exact solution of (7.9a) with \( \bar{f}(t) = 0 \) is

\[ w_h(t) = \exp(tA_h) w_h(0) = Q_h \exp(t\Lambda_h) Q_h^T w_h(0). \]

The solution remains bounded for all \( t \geq 0 \) because

\[ |w_h(t)|_h \leq \left| \exp(t\Lambda_h) \right|_h |w_h(0)|_h. \]

Does the numerical approximation have the same property?
Explicit Euler method:

\[ w_h^{n+1} = w_h^n + \tau A_h w_h^n = (I + \tau A_h) w_h^n. \]

The approximations are bounded by

\[ |w_h^n|_h \leq |I + \tau A_h|_h^n |w_h^0|_h = \max_{k=1,\ldots,m-1} |1 + \tau \lambda_{h,k}| |w_h^0|_h. \]

Hence, the numerical solution remains bounded for all \( n \in \mathbb{N} \) if

\[ |1 + \tau \lambda_{h,k}| \leq 1 \iff \tau \leq \frac{2}{|\lambda_{h,k}|} \]

for all \( k = 1, \ldots, m - 1 \). Since

\[ \lambda_{h,k} = \frac{2}{h^2} \left( \cos \left( \frac{kh\pi}{b-a} \right) - 1 \right). \]

according to Lemma 7.2.2, it follows that

\[ \max_{k=1,\ldots,m-1} |\lambda_{h,k}| \leq \frac{4}{h^2}. \]

Hence, we obtain the stability condition

\[ \tau \leq \frac{h^2}{2}. \]

This is a severe restriction, because \( h \ll 1 \) must be small to ensure an acceptable accuracy of the spatial approximation. For larger step-sizes (i.e. \( \tau > \frac{h^2}{2} \)), the norm of the numerical solution may tend to \( \infty \) whereas the exact solution remains bounded. Reducing the step size \( \tau \), however, increases the number of steps and hence the numerical costs. Inefficient!

**Definition 7.3.1 (A-stability)** The initial value problem

\[ y' = \lambda y, \quad y : [0, \infty) \to \mathbb{C}, \quad y(0) = y_0 \quad (7.13) \]

for some \( \lambda \in \mathbb{C} \) is called **Dahlquist’s test equation**. A one-step method (e.g. Runge-Kutta) is called **A-stable** if the numerical solution \((y^n)_{n \in \mathbb{N}}\) of \((7.13)\) with arbitrary \( \lambda \in \mathbb{C}, Re(\lambda) \leq 0 \) and arbitrary step-size \( \tau > 0 \) remains bounded for all \( n \in \mathbb{N} \).

**Remark.** If \( \nu_h \) is the \( k \)-th eigenvector of \( A_h \), then

\[ w_h'(t) = A_h w_h(t), \quad w_h(0) = \nu_k \]

is equivalent to \((7.13)\) with \( \lambda := \lambda_{h,k} < 0 \).

The explicit Euler method (and every other explicit Runge-Kutta method) is not A-stable.
**Implicit Euler method:** The implicit Euler

\[ w_h^{n+1} = w_h^n + \tau A_h w_h^{n+1} \quad \text{or equivalently} \quad (I - \tau A_h) w_h^{n+1} = w_h^n \]

is the simplest A-stable method. Hence, we expect that the corresponding numerical approximation of (7.9a) with \( \bar{f}(t) = 0 \) remains bounded without any step-size restrictions. This is indeed the case: Lemma 7.2.3 implies that

\[ |w_h^n|_h \leq \left| (I - \tau A_h)^{-1} \right|_h |w_h^{n-1}|_h \leq |w_h^{n-1}|_h \]

and hence \( |w_h^n|_h \leq |w_h^0|_h \) for all \( n \in \mathbb{N} \).

**Trapezoidal rule:** The trapezoidal rule is A-stable (exercise).

**Error bound for the implicit Euler method**

For simplicity we assume that \( \bar{f}(t) = \bar{f} \in \mathbb{R}^{m-1} \) is constant, i.e. we consider the initial value problem

\[ \begin{align*}
   w'(t) &= A_h w_h(t) + \bar{f} \\
   w_h(0) &= \bar{u}(0).
\end{align*} \tag{7.14a} \tag{7.14b} \]

Let \( \Psi_t : w_h(0) \mapsto w_h(t) \) be the exact flow of (7.14a). The variation-of-constants formula yields

\[ \Psi_t(y) = e^{tA_h} y + \int_0^t e^{(t-s)A_h} \bar{f} \, ds \]

for every \( y \in \mathbb{R}^{m-1} \). Define

\[ \Psi^0_t(y) = y \quad \text{and} \quad \Psi^k_t(y) = \Psi_t(\Psi^{k-1}_t(y)) = \Psi_k(y). \]

Let \( w_h^n \approx w_h(t_n) \) be the approximation computed with the implicit Euler method

\[ w_h^{n+1} = w_h^n + \tau A_h w_h^{n+1} + \tau \bar{f} \]

with \( w_h^0 = w_h(0) \). Equivalent:

\[ w_h^n = \Phi_t(w_h^{n-1}), \quad \Phi_t(y) := (I - \tau A_h)^{-1} (y + \tau \bar{f}). \]

Define

\[ \Phi^0_t(y) = y \quad \text{and} \quad \Phi^k_t(y) = \Phi_t(\Phi^{k-1}_t(y)). \]

Remark: \( \Psi_t \) and \( \Phi_t \) depend on \( h \), of course, but for simplicity we do not express this in the notation.
Theorem 7.3.2 (error bound for the implicit Euler method) For every \( h > 0 \) and \( N \in \mathbb{N} \) the error of the implicit Euler method applied to (7.14) with step-size \( \tau = t_{\text{end}}/N \) is bounded by

\[
\max_{n=0, \ldots, N} |w_n(t) - w^n_h|_h \leq C_h t_{\text{end}} \tau \tag{7.15}
\]

with constant \( C_h = \frac{3}{2} \max_{t \in [0, t_{\text{end}}]} |w''_n(t)|_h \).

Proof.

Step 1: Local error. Prove a bound for the error after one step starting with exact data \( w_h(t_j) \), i.e. for \( |\Psi_\tau(w_h(t_j)) - \Phi_\tau(w_h(t_j))|_h \) with arbitrary \( j = 0, \ldots, N - 1 \).

Expansion of the exact flow:

\[
\Psi_\tau(w_h(t_j)) = w_h(t_j + \tau) = w_h(t_j) + \tau w'_h(t_j) + \frac{\tau^2}{2} w''_h(\xi) \quad \xi \in (t_j, t_j + \tau)
\]

\[
= (I + \tau A_h) w_h(t_j) + \tau f + \frac{\tau^2}{2} w''_h(\xi). \tag{7.16}
\]

Use that \( (I - \tau A_h)^{-1} (I - \tau A_h) = I \) implies

\[
(I - \tau A_h)^{-1} = I + \tau (I - \tau A_h)^{-1} A_h = I + \tau A_h + \tau^2 (I - \tau A_h)^{-1} A_h^2.
\]

This yields the following expansion of the numerical flow

\[
\Phi_\tau(w_h(t_j)) = (I - \tau A_h)^{-1} (w_h(t_j) + \tau \bar{f}) = (I + \tau A_h + \tau^2 (I - \tau A_h)^{-1} A_h^2) w_h(t_j) + \tau (I + \tau (I - \tau A_h)^{-1} A_h) \bar{f} = (I + \tau A_h) w_h(t_j) + \tau \bar{f} + \tau^2 (I - \tau A_h)^{-1} (A_h^2 w_h(t_j) + A_h \bar{f}) \tag{7.17}
\]

because \( w''_n(t) = \bar{A}_h^2 w_n(t) + A_h \bar{f} \). Comparing (7.16) with (7.17) and applying Lemma 7.2.3 shows that

\[
|\Psi_\tau(w_h(t_j)) - \Phi_\tau(w_h(t_j))|_h \leq \frac{3}{2} \max_{t \in [0, t_{\text{end}}]} |w''_n(t)|_h \tau^2 = C_h \tau^2.
\]

Step 2: Stability. By definition of \( \Phi_\tau(y) = (I - \tau A_h)^{-1} (y + \tau \bar{f}) \) we have for every \( y \) and \( \tilde{y} \)

\[
|\Phi_\tau(y) - \Phi_\tau(\tilde{y})|_h = |(I - \tau A_h)^{-1}(y - \tilde{y})|_h \leq |y - \tilde{y}|_h
\]

and by induction

\[
|\Phi^k_\tau(y) - \Phi^k_\tau(\tilde{y})|_h \leq |y - \tilde{y}|_h
\]
Step 3: Error accumulation and global error. We represent the global error by the telescoping sum (“Lady Windermere’s fan”)

\[
    w_h(t_n) - w^m_h = \Psi^0_\tau(w^0_h) - \Phi^0_\tau(w^0_h) = \sum_{k=0}^{n-1} \left( \Phi^k_\tau(\Psi^{n-k}(w^0_h)) - \Phi^{k+1}_\tau(\Psi^{n-k-1}(w^0_h)) \right).
\]

(7.18)

Figure 7.1: Lady Windermere’s fan. To improve readability the subscript \( h \) has been omitted everywhere.

From step 1 and 2, we conclude that

\[
    \left| \Phi^k_\tau(\Psi^{n-k}(w^0_h)) - \Phi^{k+1}_\tau(\Psi^{n-k-1}(w^0_h)) \right|_h \leq \left| \Psi^{n-k}(w^0_h) - \Phi^{n-k-1}_\tau(\Psi^{n-k-1}(w^0_h)) \right|_h = \left| \Psi^{n-k}(w^0_h) \right|_h = \left| \Psi^{n-k-1}(w^0_h) \right|_h \leq C_h \tau^2.
\]

Taking norms in (7.18) and applying the triangle inequality thus gives

\[
    |w_h(t_n) - w^m_h|_h \leq \tau^2 \sum_{k=0}^{n-1} C_h = \tau \sum_{k=0}^{n-1} C_h \leq C_h t_{\text{max}} \tau.
\]

General principle: Consistency + stability \( \implies \) convergence
7.4 Full discretization of the heat equation in time and space

Let $u$ be the solution of the heat equation (7.8) with $f(t, x) = f(x)$ independent of $t$, and let

$$\bar{u}(t) = \left( u(t, x_1), \ldots, u(t, x_{m-1}) \right)^T \in \mathbb{R}^{m-1}$$

(7.19)

$$\bar{f} = \left( f(x_1), \ldots, f(x_{m-1}) \right)^T \in \mathbb{R}^{m-1}$$

(7.20)

contain the values at the inner grid points. Let $w_h(t) \approx \bar{u}(t)$ be the solution of the spatially discretized problem (7.9), and let $w_{h n} \approx w_h(t_n) \approx \bar{u}(t_n)$ the approximation computed with the implicit Euler method.

**Corollary 7.4.1** Assume that

$$u(t, \cdot) \in C_4^4([a, b])$$

for every $t \in [0, t_{\text{end}}]$, (7.21)

$$\sup_{t \in [0, t_{\text{end}}]} \| \partial_x^4 u(t, \cdot) \|_{\infty} < \infty$$

(7.22)

as in Theorem 7.2.5. Moreover, assume that

$$\max_{t \in [0, t_{\text{end}}]} |w''_h(t)|_h \leq \hat{C}$$

(7.23)

with a constant $\hat{C}$ which does not depend on $h$. Then, the error of the full discretization at $t_n$ is bounded by

$$\max_{n=0, \ldots, N} |\bar{u}(t_n) - w_{h n}|_h \leq C t_{\text{end}} (h^2 + \tau)$$

with a constant $C$ which does not depend on $h$. If (7.21), (7.22) and

$$\partial_x^2 u(0, a) = \partial_x^2 u(0, b) = 0, \quad f \in C_2^2([a, b])$$

(7.24)

then (7.23) is true.

**Proof.** Theorem 7.2.5 and Theorem 7.3.2 yield

$$|\bar{u}(t_n) - w_{h n}|_h \leq |\bar{u}(t_n) - w_h(t_n)|_h + |w_h(t_n) - w_{h n}|_h$$

$$\leq C t_{\text{end}} h^2 \sup_{t \in [0, t_{\text{end}}]} \| \partial_x^4 u(t, \cdot) \|_{\infty} + C h t_{\text{end}} \tau$$

for every $n = 0, \ldots, N$. Since

$$C_h = \frac{3}{2} \max_{t \in [0, t_{\text{end}}]} |w''_h(t)|_h \leq \frac{3\hat{C}}{2}$$
can be bounded independently of $h$ by assumption, the first assertion follows.

Now we show that (7.21), (7.22), and (7.24) imply (7.23). Taking the time derivative of the ODE

$$w'_h(t) = A_h w_h(t) + \bar{f}$$

and substituting $w_h(t)$ by the variation-of-constants formula yields

$$w''_h(t) = A_h^2 w_h(t) + A_h \bar{f}$$

$$= A_h^2 \left( e^{tA_h} \bar{u}(0) + \int_0^t e^{(t-s)A_h} \bar{f} \, ds \right) + A_h \bar{f}$$

$$= e^{tA_h} A_h^2 \bar{u}(0) - \left[ e^{(t-s)A_h} A_h \bar{f} \right]_{s=0}^t + A_h \bar{f}$$

$$= e^{tA_h} A_h^2 \bar{u}(0) + e^{tA_h} A_h \bar{f}.$$

Since $|e^{tA_h}|_h = |e^{tA_h}| \leq 1$ by Lemma 7.2.3, we obtain

$$|w''_h(t)|_h \leq |A_h^2 \bar{u}(0)|_h + |A_h \bar{f}|_h.$$

It follows from Lemma 7.2.1 that

$$|A_h \bar{u}(0) - \partial_x^2 u(0)|_h \leq C h^2, \quad \partial_x^2 u(0) = \left( \partial_x^2 u(0, x_1), \ldots, \partial_x^2 u(0, x_m-1) \right)^T$$

with $C$ independent of $h$. Moreover, it can be shown by Taylor expansions that for every function $v \in C^2_0([a, b])$ we have

$$|A_h \bar{v}|_h \leq \sqrt{b-a} \parallel v'' \parallel_\infty, \quad \bar{v} = \left( v(x_1), \ldots, v(x_m-1) \right)^T. \quad (7.25)$$

This yields

$$|w''_h(t)|_h \leq \left| A_h (A_h \bar{u}(0) - \partial_x^2 u(0)) \right|_h + \left| A_h \partial_x^2 u(0) \right|_h + \left| A_h \bar{f} \right|_h$$

$$\leq |A_h|h C h^2 + \sqrt{b-a} \parallel \partial_x^2 u(0, \cdot) \parallel_\infty + \sqrt{b-a} \parallel f'' \parallel_\infty \leq \tilde{C}$$

because $|A_h|h \leq 4/h^2$ according to Lemma 7.2.2.

If the initial value problem for $w_h(t)$ is solved with the trapezoidal rule instead of the implicit Euler and slightly stronger regularity assumptions are made, then we obtain the error bound

$$\max_{n=1,\ldots,N} |\bar{u}(t_n) - w''_h|_h \leq c t_{end}(h^2 + \tau^2)$$
because the order of the trapezoidal rule is 2. The combination of the trapezoidal rule in time and finite differences in space is called Crank-Nicolson method in the literature. The results obtained with the implicit Euler method, however, are sometimes better because this method is L-stable such that errors committed in previous steps are damped; cf. IV.3 in [HW10]

If a Runge-Kutta method of order $p$ is applied, then one would expect a total error of $ct_{	ext{end}}(h^2 + \tau^p)$. Unfortunately, this is in general not the case! The reason is that the error analysis for ODEs is based on regularity assumptions of the right-hand side which are not given in case of the heat equation. Roughly speaking, this is due to the fact that for $m \to \infty$ we have $|A_n| \to \infty$, and the problem becomes “infinitely stiff”. This leads to order reduction; cf. IV.15 in [HW10].

**Solving the linear systems**

A disadvantage of implicit Runge-Kutta methods is the fact that in each step a system of (typically nonlinear) equations has to be solved. In case of the heat equation, this system is linear. If the implicit Euler method is applied to the spatially discretized problem

\[
\begin{align*}
w'_h(t) & = A_h w_h(t) + \bar{f} \\
w_h(0) & = \bar{u}(0),
\end{align*}
\]

then we have to solve

\[
M w_h^{n+1} = w_h^n + \tau \bar{f}(t) =: z \quad \text{with} \quad M := I - \tau A_h.
\]

For arbitrary $\tau$ and $h$ the matrix $M$ is symmetric and positive definite. Hence, the linear system can be solved via the Cholesky decomposition $M = LL^T$. It can be shown that the Cholesky decomposition does not produce any fill-in, i.e. $L_{jk} = 0$ for $j \notin \{k, k + 1\}$. As a consequence, the numerical work for solving the linear system is only $O(m)$ instead of $O(m^3)$.

### 7.5 Application to the Black-Scholes equation

Consider a European capped symmetric power call, i.e. a European call option with payoff

\[
V(T, S) = \min \left( L, ((S - K)^+)^p \right)
\]

and maturity $T > 0$, strike $K > 0$ and parameters $p, L > 0$.

- “capped”: $V(T, S) \leq L$ bounded
  - “power”: $((S - K)^+)^p$ instead of $(S - K)^+$
  - “symmetric”: $((S - K)^+)^p$ instead of $(S^p - K)^+$
For $p = 1$ and $L = \infty$: standard European call

The value $V(t, S)$ of the option evolves according to the Black-Scholes equation

$$
\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial^2_S V(t, S) + rS \partial_S V(t, S) - rV(t, S) = 0, \quad S \in (0, \infty), \ t \in [0, T]
$$

with volatility $\sigma > 0$, interest rate $r > 0$ and terminal condition (7.26). At the left boundary $S = 0$, no boundary condition is required, because the two terms $S^2 \partial^2_S V(t, S)$ and $rS \partial_S V(t, S)$ vanish for $S \to 0$ if the solution is sufficiently smooth; cf. 3.2. For $S = 0$ the PDE reduces to the ODE

$$
\partial_t V(t, 0) - rV(t, 0) = 0,
$$

and since the terminal condition (7.26) implies $V(T, 0) = 0$, it follows that

$$
V(t, 0) = 0 \quad \text{for all } t \in [0, T].
$$

(a) Truncation of the domain

Replace $(0, \infty)$ by $(0, S_{\text{max}})$ with a sufficiently large $S_{\text{max}} > K + L^{1/p}$ and boundary condition

$$
V(t, S_{\text{max}}) = L \quad \text{for all } t \in [0, T]
$$

(b) Time inversion and homogeneous boundary data

Let $f(S) = \frac{L}{S_{\text{max}}} S$ and let $u(t, S) := V(T - t, S) - f(S)$. Then, the problem reads (check!)

$$
\partial_t u = \frac{\sigma^2}{2} S^2 \partial^2_S u + rS \partial_S u - ru, \quad S \in (0, S_{\text{max}}), \ t \in (0, T]
$$

$$
u(0, S) = \min \left( L, \left( (S - K)^+ \right)^p \right) - f(S), \quad S \in [0, S_{\text{max}}]
$$

$$
u(t, 0) = u(t, S_{\text{max}}) = 0, \quad t \in [0, T]
$$

The boundary “condition” at $S = 0$ is automatically fulfilled.

(c) Space discretization

Approximation of derivatives by finite differences yields the ODE

$$
\bar{w}_h'(t) = M_h w_h(t), \quad t \in (0, t_{\text{end}}]
$$

$$
\bar{w}_h(0) = \bar{u}(0)
$$

with

$$
\bar{u}(t) = \left( u(t, S_1), \ldots, u(t, S_{m-1}) \right)^T;
$$

see exercise class for details. The solution yields an approximation $w_h(t) \approx \bar{u}(t)$. 
(d) Time discretization

Solve the ODE with the implicit Euler method

\[(I - \tau M_h)w_{h}^{n+1} = w_{h}^{n}\]

or the trapezoidal rule

\[(I - \tau M_h/2)w_{h}^{n+1} = (I + \tau M_h/2)w_{h}^{n}.

Final result: \(w_{h}^{n} \approx w_h(t_n) \approx \bar{u}(t_n), \) i.e. \(V(T - t_n, S_k) \approx V_{h,k}^{N-n} := w_{h,k}^{n} + f(S_k).\)

(e) Numerical experiments

See slides. The numerical examples show that the expected order of convergence is not achieved in practice because the payoff function (i.e. the initial data) does not have the required \(C^4\) regularity. Nevertheless, we still observe convergence of the methods at some lower order. This calls for an explanation.

7.6 Non-smooth initial data

(a) Parabolic smoothing

For the our analysis, we consider again the model problem (7.8), i.e. the heat equation on an interval with homogeneous Dirichlet boundary conditions. For simplicity, we assume that \(f(t, x) = 0\) and that \((a, b) = (0, \pi):\)

\[\partial_t u(t, x) = \partial_x^2 u(t, x) \quad t \in (0, t_{\text{end}}], \ x \in (0, \pi) \quad (7.27a)\]

\[u(t, 0) = u(t, \pi) = 0 \quad t \in [0, t_{\text{end}}] \quad (7.27b)\]

\[u(0, x) = u_0(x) \quad x \in [0, \pi) \quad (7.27c)\]

**Theorem 7.6.1 (Solution of the heat equation)** If \(u_0\) is continuous and piecewise continuously differentiable with finitely many “kinks”, then the unique solution of (7.27a)-(7.27c) is given by the Fourier series

\[u(t, x) = \sum_{k=1}^{\infty} c_k \sin(kx)e^{-k^2t} \quad \text{with} \quad c_k = \frac{2}{\pi} \int_0^\pi u_0(x) \sin(kx) \, dx. \quad (7.28)\]

For the proof we need the following classical result.

**Lemma 7.6.2 (Maximum principle)** Let \(\Omega := \{(t, x) \mid 0 < x < \pi, \ t > 0\}\) and \(\{t \leq t_{\text{end}}\} := \{(t, x) \mid t \leq t_{\text{end}}\}\). Then, every continuous solution on \(\Omega\) of the heat equation (7.27) has the property that

\[\min_{\{t \leq t_{\text{end}}\} \cap \partial \Omega} u \leq u(t, x) \leq \max_{\{t \leq t_{\text{end}}\} \cap \partial \Omega} u\]

for all \((t, x) \in \{t \leq t_{\text{end}}\} \cap \Omega.\)
Proof of Theorem 7.6.1.
• PDE and boundary conditions: Each term of the Fourier series solves (7.27a) and (7.27b):
\[
\partial_t \left( c_k \sin(kx)e^{-k^2t} \right) = -k^2 c_k \sin(kx)e^{-k^2t} = \partial_x^2 \left( c_k \sin(kx)e^{-k^2t} \right)
\]
\[
c_k \sin(kx)e^{-k^2t} = 0 \quad \text{for } x \in \{0, \pi\}.
\]
• Orthogonality of \(\{\sin(jx), j \in \mathbb{N}_0\}\) in \(L^2([0, \pi])\):
\[
\int_0^\pi \sin(jx) \sin(kx) \, dx = \begin{cases} 
\pi & \text{if } j = k \\
\frac{\pi}{2} & \text{if } j \neq k.
\end{cases}
\]
• Initial data:
\[
u(t,x) = \begin{cases} 
u_0(x) & \\
\int_0^\pi \nu(t,x) \sin(kx) \, dx = \int_0^\pi \nu_0(x) \sin(kx) \, dx & \text{for all } k \in \mathbb{N}_0.
\end{cases}
\]
\[
\sum_{j=1}^\infty c_j \int_0^\pi \sin(jx) \sin(kx) \, dx = \int_0^\pi \nu_0(x) \sin(kx) \, dx & \text{for all } k \in \mathbb{N}_0
\]
\[
\pi \frac{c_k}{2} \int_0^\pi \nu_0(x) \sin(kx) \, dx & \text{for all } k \in \mathbb{N}_0.
\]
• Uniqueness: Suppose that \(u^1\) and \(u^2\) are two solutions of (7.27). Then the difference \(\tilde{u} = u^1 - u^2\) is also a solution of (7.27a) and (7.27b) with initial data \(\tilde{u}(0,x) = 0\) for all \(x \in [0, \pi]\). Hence,
\[
\min_{\{t \leq t_{\text{end}}\} \cap \partial \Omega} \tilde{u} = \max_{\{t \leq t_{\text{end}}\} \cap \partial \Omega} \tilde{u} = 0 \implies \tilde{u} = 0 \quad \text{for all } (t,x) \in \{t \leq t_{\text{end}}\} \cap \Omega
\]
\[
\implies u^1 = u^2
\]
• It remains to show that derivatives of the series (7.28) are obtained by deriving term by term. Under the regularity assumptions on \(\nu_0\) it can be shown that
\[
\sum_{k=1}^\infty |c_k| < \infty;
\]
(7.29)
\[
\text{cf. \S 6, 2.8 in volume 2 of [FK08]. Hence, } \lim_{k \to \infty} c_k = 0, \text{ and } \hat{c} := \sup_{k \in \mathbb{N}} |c_k| < \infty \text{ exists.}
\]
Choose a fixed \(t > 0\). Since the Fourier series
\[
u(t,x) = \sum_{k=1}^\infty c_k \sin(kx)e^{-k^2t}
\]
is dominated by (7.29), it converges uniformly. Hence, $u(t, \cdot)$ is continuous. For every fixed $p \in \mathbb{N}$ and every $t > 0$ the series

$$
\sum_{k=1}^{\infty} k^p e^{-k^2 t}
$$

(7.30)

converges by the ratio test, because

$$
\frac{|(k + 1)^p e^{-(k+1)^2 t}|}{|k^p e^{-k^2 t}|} = \left(\frac{k + 1}{k}\right)^p e^{-(k+1)^2 t} = \left(\frac{k + 1}{k}\right)^p e^{(2k+1) t} \rightarrow 0.
$$

Since $|k c_k e^{-k^2 t} \cos(kx)| \leq \hat{c}|ke^{-k^2 t}|$, the series

$$
\sum_{k=1}^{\infty} k c_k e^{-k^2 t} \cos(kx)
$$

(7.31)

converges uniformly. This means that (7.31) coincides with $\partial_x u(t, x)$. In a similar way, we obtain from (7.30) with $p = 2$ that

$$
\partial_t u(t, x) = -\sum_{k=1}^{\infty} k^2 c_k e^{-k^2 t} \sin(kx) = \partial_x^2 u(t, x).
$$

Remarks:

1. The function

$$
\tilde{u}(t, x) = v_0 + \frac{v_\pi - v_0}{\pi} x + u(t, x)
$$

solves (7.27a) with nonzero boundary conditions $\tilde{u}(t, 0) = v_0$ and $\tilde{u}(t, \pi) = v_\pi$. Solutions on arbitrary intervals can be constructed by rescaling.

2. Since (7.1) involves $\partial_x^2 u(t, x)$, one may expect that all solutions of the PDE are twice continuously differentiable with respect to $x$. Theorem 7.6.1 shows, however, that solutions with lower regularity of the initial data exist.

3. All terms in the series representation (7.28) oscillate in space due to the term $\sin(kx)$, and the higher $k$, the faster the oscillations. The $k$-th term, however, is multiplied with $c_k e^{-k^2 t}$, i.e. it decays exponentially as time evolves, and the larger $k$, the faster the decay. This has a surprising consequence:
Theorem 7.6.3 (Parabolic smoothing) Let $u_0$ be continuous and piecewise continuously differentiable with finitely many “kinks”, and let $u(t,x)$ be the solution of (7.27a)-(7.27c). Then

$$x \mapsto u(t,x) \in C^\infty(0,\pi)$$

for every $t > 0$.

This means that after an arbitrarily short time the solution is infinitely smooth although the initial data may have a much lower regularity. Hence, only $t \approx 0$ is critical for the approximation with finite differences.

Proof. For any $q \in \mathbb{N}$, the argument in the proof of Theorem 7.6.1 can be applied to

$$\pm \sum_{k=1}^{\infty} k^{2q+1} c_k e^{-k^2 t} \cos(kx)$$

and

$$\pm \sum_{k=1}^{\infty} k^{2q} c_k e^{-k^2 t} \sin(kx).$$

This yields the assertion.

(b) Alternative error bound for the implicit Euler method

After discretizing (7.27) in space with finite differences, we obtain the ODE

$$w'_h(t) = A_h w_h(t), \quad w_h(0) = \bar{u}(0).$$

with $A_h \in \mathbb{R}^{(m-1) \times (m-1)}$ defined in (7.4) and $h = \pi/m$. Let $w^n_h \approx w(t_n)$ be the approximation given by the implicit Euler method:

$$w^{n+1}_h = w^n_h + \tau A_h w^{n+1}_h, \quad w^0_h = w_h(0)$$

with $N \in \mathbb{N}$, $\tau = t_{\text{end}}/N$ and $n = 0, \ldots, N - 1$.

Theorem 7.6.4 For all $n = 1, \ldots, N$, the error of the implicit Euler method is bounded by

$$|w_h(t_n) - w^n_h|_h \leq C\frac{\tau}{t_n} |w_h(0)|_h$$

(7.32)

with a constant $C \geq 0$ which does not depend on $m$, $n$ or $\tau$.

Interpretation. This result differs from the corresponding error bounds in sections 7.3 and 7.4 in several ways. First, the term $1/t_n$ is new. This term means that for small $n$ (i.e. in the first steps), the error can be very large, but it also means that for large $t_n$ the error will vanish. This is exactly what we have observed in the numerical simulations of the power option in section 7.5 (e). Moreover, the error does not depend on the term

$$\max_{t \in [0,t_{\text{end}}]} |w^n_h(t)|_h.$$  

(7.33)

In Corollary 7.4.1, we had to make the regularity assumptions (7.21), (7.22), (7.24) in order to prove that (7.33) is bounded uniformly in $h$.

A more general result is shown in [Tho06], Theorem 7.2, p. 117.
**Numerical illustration:** See slides. The numerical experiment is done on \([0, 2]\) instead of \([0, \pi]\), but this does not change the assertions of Theorem 7.6.4.

**Proof.** As in the proof of Lemma 7.2.3 we consider the diagonalization \(A_h = Q_h \Lambda_h Q_h^T\) with

\[
\Lambda_h = \text{diag}(\lambda_{h,1}, \ldots, \lambda_{h,m-1}), \quad \lambda_{h,k} = \frac{2}{h^2} \left( \cos \left( \frac{kh\pi}{b-a} \right) - 1 \right) < 0.
\]

Since \(Q_h^T Q_h = I\) the function \(y_h(t) := Q_h^T w_h(t)\) solves the ODE

\[
y_h'(t) = Q_h^T w_h'(t) = Q_h^T Q_h \Lambda_h Q_h^T w_h(t) = \Lambda_h y_h(t),
\]

and hence \(y_h(t) = e^{t \Lambda_h} y_h(0)\). In a similar way, we transform the approximations \(w_h^n\) of the implicit Euler method: If we let \(y_h^n = Q_h^T w_h^n\), then

\[
y_h^n = y_h^{n-1} + \Lambda_h y_h^n, \quad y_h^n = (I - \tau \Lambda_h)^{-1} y_h^{n-1} = (I - \tau \Lambda_h)^{-n} y_h^0.
\]

Hence, we only have to show that

\[
|e^{n\mu} - (1 - \tau \lambda)^{-n}| \leq C \frac{\tau}{t_n} \quad \text{for all } \lambda < 0 \text{ and } n = 1, \ldots, N.
\]

or, with \(\mu = \tau \lambda, \tau/t_n = 1/n\) and \(R(\mu) = \frac{1}{1-\mu}\), that

\[
|e^{n\mu} - R^n(\mu)| \leq \frac{C}{n} \quad \text{for all } \mu \in (-\infty, 0) \text{ and } n = 1, \ldots, N. \tag{7.34}
\]

**Case 1: \(\mu \in [-1, 0]\).** Comparing the Taylor expansion

\[
e^\mu = 1 + \mu + \int_0^{\mu} \int_0^s e^{-r} \, dr \, ds
\]

with

\[
R(\mu) = 1 + \mu + \frac{\mu^2}{1 - \mu}
\]

yields for all \(\mu \in [-1, 0]\) the “local” error bound

\[
|e^{n\mu} - R^n(\mu)| \leq \int_0^{\mu} \int_0^s |e^{-r}| \, dr \, ds + \frac{\mu^2}{1 - \mu}
\]

\[
\leq \int_0^{\mu} \int_0^s 1 \, dr \, ds + \mu^2 = \frac{3}{2} \mu^2. \tag{7.35}
\]
Moreover, it can be shown that \(^1\)

\[
R(\mu) = \frac{1}{1-\mu} \leq e^{\gamma \mu} \quad \text{for} \quad \gamma := \log(2) \quad \text{and} \quad \mu \in [-1, 0)
\]  

(7.36)

Substituting (7.35) and (7.36) into the telescoping sum

\[
e^{n\mu} - R^n(\mu) = \sum_{k=0}^{n-1} R(\mu)^{n-1-k} (e^{\mu} - R(\mu)) e^{k\mu}
\]

yields

\[
|e^{n\mu} - R^n(\mu)| \leq \sum_{k=0}^{n-1} R^{n-1-k}(\mu) |e^{\mu} - R(\mu)| e^{k\mu}
\]

\[
\leq \frac{3}{2} \mu^2 \sum_{k=0}^{n-1} e^{\gamma \mu (n-1-k)} e^{k\mu}
\]

\[
\leq \frac{3}{2} \mu^2 e^{\gamma \mu (n-1)} \sum_{k=0}^{n-1} e^{k\mu (1-\gamma)}.
\]

Since \(1 - \gamma = 1 - \log(2) > 0\) and \(\mu < 0\), it follows that

\[
\sum_{k=0}^{n-1} e^{k\mu (1-\gamma)} \leq \sum_{k=0}^{n-1} 1 = n
\]

and hence for \(\mu \in [-1, 0)\)

\[
|e^{n\mu} - R^n(\mu)| \leq \frac{3}{2} n \mu^2 e^{\gamma \mu n} e^{-\gamma \mu} \leq e^\gamma = \frac{3}{2} (\mu n)^2 e^{\gamma \mu n} e^{-\gamma} \frac{1}{n}. \leq C
\]

\(^1\)Since \((1-\mu) - 1 > 0\) it is true that

\[
R(\mu) = \frac{1}{1-\mu} \leq e^{\gamma \mu} \iff 0 \leq f(\mu) := e^{\gamma \mu} (1-\mu) - 1.
\]

Since

\[
f(0) = 1 - 1 = 0 \quad \text{and} \quad f(-1) = e^{-\log(2)} 2 - 1 = 2^{-1} \cdot 2 - 1 = 0,
\]

it is enough to show that \(f\) has a maximum but no minimum in \((-1, 0)\). This follows from

\[
f'(\mu) = e^{\gamma \mu} (\gamma - \gamma \mu - 1) = 0 \iff (\gamma - \gamma \mu - 1) = 0 \iff \mu = \frac{\gamma - 1}{\gamma}
\]

\[
f\left(\frac{\gamma - 1}{\gamma}\right) = e^{\gamma - 1} \left(1 - \frac{\gamma - 1}{\gamma}\right) - 1 = \frac{e^{\gamma - 1}}{\gamma} - 1 = \frac{2e^{-1}}{\gamma} - 1 \approx 0.0615 > 0.
\]
Case 2: $\mu < -1$. In this case, we simply use the triangular inequality

$$|e^{n\mu} - R^n(\mu)| \leq |e^{n\mu}| + |R^n(\mu)|$$

and bound both terms separately. It follows from $e^n \geq n + 1 > n$ that $e^{n\mu} < e^{-n} < \frac{1}{n}$. Moreover,

$$R(\mu) = \frac{1}{1 - \mu} < \frac{1}{2} = e^{-\gamma} \quad \implies \quad R^n(\mu) \leq e^{-n\gamma} < e^{-n} \leq \frac{1}{n}$$

and hence $|e^{n\mu} - R^n(\mu)| < \frac{2}{n}$. Both cases together prove (7.34) and hence (7.32).
Chapter 8

Finite-difference methods for American options

8.1 Modelling American options

American options can be exercised before maturity. Mathematical model?

Properties of American options: If $V_{\text{Am}}^C(t, S)$ is the value of an American call, $V_{\text{Eu}}^P(t, S)$ is the value of an European put, etc., then

\[
(K - S)^+ \leq V_{\text{Am}}^P(t, S) \leq K \tag{8.1}
\]

\[
V_{\text{Eu}}^P(t, S) \leq V_{\text{Am}}^P(t, S)
\]

$V_{\text{Eu}}^C(t, S) = V_{\text{Am}}^C(t, S)$ if no dividends are paid

$V_{\text{Eu}}^C(t, S) \leq V_{\text{Am}}^C(t, S)$ if dividends are paid

Proof: Proposition 2.7 and remark 2.9 in [GJ10].

In the entire chapter, we consider American puts with no dividends and drop the indices, i.e. $V(t, S) = V_{\text{Am}}^P(t, S)$. American calls (with dividends) can be treated in a similar way.

For every $t \in [0, T)$, there is a unique $0 \leq S_*(t) < K$ such that

\[
V(t, S) > (K - S)^+ \quad \text{for } S > S_*(t) \quad \Longrightarrow \text{ no early exercise } \tag{8.2a}
\]

\[
V(t, S) = (K - S)^+ \quad \text{for } S \leq S_*(t) \quad \Longrightarrow \text{ early exercise } \tag{8.2b}
\]

Sketch of the proof. For $S = 0$ the inequalities (8.1) imply

\[
V(t, 0) = K = (K - 0)^+.
\]

Since $V(t, S) > 0$ for $t < T$, it follows that

\[
V(t, S) > 0 = (K - S)^+ \quad \text{for } S \geq K.
\]
A monotonicity argument yields the existence and uniqueness of $0 \leq S_*(t) < K$. 

For fixed $t$, $S_*(t)$ is called the **contact point**, and the function $t \mapsto S_*(t)$ is called the **early-exercise curve**, because the option should not be exercised at time $t < T$ if $S > S_*(t)$.

For $S \leq S_*(t)$, the value of the option is known. For $S > S_*(t)$ the option is not exercised and can thus be modeled by the Black-Scholes equation

$$
\partial_t V + \mathcal{A} V = 0, \quad \mathcal{A} V = \frac{\sigma^2}{2} S^2 \partial^2_S V + r S \partial_S V - r V.
$$

If $S \mapsto V(t, S)$ is $C^1$ for all $t \in [0, T)$, then it follows that

$$
\partial_S V(t, S_*(t)) = -1.
$$

Hence, we have to solve the **free boundary value problem**

$$
\begin{align*}
\partial_t V(t, S) + \mathcal{A} V(t, S) & = 0 \quad \text{for } S > S_*(t), \ t \in [0, T) \quad (\text{PDE}) \quad (8.3a) \\
V(T, S) & = (K - S)^+ \quad \text{for } S \geq 0 \quad (\text{terminal cond.}) \quad (8.3b) \\
V(t, S_*(t)) & = (K - S_*(t))^+ \quad \text{for } t \in [0, T) \quad (\text{Dirichlet b.c.}) \quad (8.3c) \\
\partial_S V(t, S_*(t)) & = -1 \quad \text{for } t \in [0, T) \quad (\text{Neumann b.c.}) \quad (8.3d)
\end{align*}
$$

**Problem:** The boundary $S_*(t)$ changes in time and depends on the solution. **Goal:** Reformulate the problem without $S_*(t)$.

We know that for $S \leq S_*(t) < K$, the value of the option is $V(t, S) = (K - S)^+ = K - S =: V_*(S)$. The function $V_*(S)$, however, does **not** solve the Black-Scholes equation (8.3a), because

$$
\partial_t V_*(S) + \mathcal{A} V_*(S) = \partial_t (K - S) + \frac{\sigma^2}{2} S^2 \partial^2_S (K - S) + r S \partial_S (K - S) - r (K - S) = -r K < 0.
$$

However, the function $V_*(S) = K - S$ satisfies the **Black-Scholes inequality**

$$
\partial_t V(t, S) + \mathcal{A} V(t, S) \leq 0 \quad \text{for } S \geq 0, \ t \in [0, T),
$$

and we know:

$$
\begin{align*}
S > S_*(t) \iff V(t, S) > (K - S)^+ \iff \partial_t V(t, S) + \mathcal{A} V(t, S) = 0 \\
S \leq S_*(t) \iff V(t, S) = (K - S)^+ \iff \partial_t V(t, S) + \mathcal{A} V(t, S) < 0
\end{align*}
$$
Hence, the free boundary value problem (8.3) is equivalent to the linear complementary problem
\[
(V(t, S) - (K - S)^+) \left( \partial_t V(t, S) + AV(t, S) \right) = 0 \quad S > 0, t \in [0, T) \tag{8.4a}
\]
\[-(\partial_t V(t, S) + AV(t, S)) \geq 0 \tag{8.4b}
\]
\[V(t, S) - (K - S)^+ \geq 0 \tag{8.4c}
\]
with terminal condition
\[V(T, S) = (K - S)^+, \quad S \in [0, \infty) \tag{8.4d}
\]
and boundary condition
\[V(t, 0) = K, \quad t \in [0, T]. \tag{8.4e}
\]

8.2 Discretization

Proceed as in Section 7.5.

(a) Truncation of the domain

Choose sufficiently large $S_{\text{max}} > K$ and consider $S \in (0, S_{\text{max}})$ instead of $S \in (0, \infty)$. Let $\tilde{V}$ be the solution on $(0, S_{\text{max}})$ with boundary condition $\tilde{V}(t, S_{\text{max}}) = 0$ for all $t \in [0, T]$.

(b) Time inversion and homogeneous boundary data

Define
\[f(S) = \left(1 - \frac{S}{S_{\text{max}}}\right) K \quad \text{and} \quad u(t, S) = \tilde{V}(T - t, S) - f(S).
\]
and observe that
\[Af(S) = \frac{\sigma^2}{2} S^2 f''(S) + rS f'(S) - rf(S) = -rK.
\]
Hence, $u(t, S)$ is the solution of the linear complementary problem
\[
(u(t, S) - u(0, S)) \left( \partial_t u(t, S) - Au(t, S) + rK \right) = 0, \quad S \in (0, S_{\text{max}}), \quad t \in (0, T], \tag{8.5a}
\]
\[\partial_t u(t, S) - Au(t, S) + rK \geq 0, \quad t \in (0, T], \tag{8.5b}
\]
\[u(t, S) - u(0, S) \geq 0 \tag{8.5c}
\]
with initial condition
\[u(0, S) = (K - S)^+ - f(S), \quad S \in [0, \infty)
\]
and boundary conditions
\[u(t, 0) = K - f(0) = 0, \tag{8.5d}
\]
\[u(t, S_{\text{max}}) = 0 - f(S_{\text{max}}) = 0, \quad t \in [0, T]. \tag{8.5e}
\]
(c) Time and space discretization

Choose $1 < m \in \mathbb{N}$, define mesh width $h = S_{\text{max}}/m$ and grid points $S_k = kh$.
Choose $N \in \mathbb{N}$, define step-size $\tau = t_{\text{end}}/N$ and $t_n = n\tau$.
Goal: Compute approximation $w^n = (w^n_1, \ldots, w^n_{m-1})^T$, $w^n_k \approx u(t_n, S_k)$.

Discretizing the PDE
\[
\partial_t u(t, S) = A u(t, S) - rK, \quad t \in (0, T], S \in (0, S_{\text{max}})
\]
\[
u(t, S) = u(t, S_{\text{max}}) = 0
\]
\[
u(0, S) = (K - S)^+ - f(S)
\]
with the Crank-Nicolson scheme (finite differences in space, trapezoidal rule in time) yields
\[
w^{n+1} = w^n + \frac{\tau}{2} M_h (w^{n+1} + w^n) - \tau rK
\]
\[
w^0 = (u(0, S_1), \ldots, u(0, S_{m-1}))^T
\]
with
\[
M_h = \frac{\sigma^2}{2} D^2 A_h + r DB_h - r I
\]
\[
D = \text{diag}(S_1, \ldots, S_{m-1}) \in \mathbb{R}^{(m-1) \times (m-1)}
\]
\[
B_h = \frac{1}{2h} \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & -1 & 0
\end{pmatrix} \in \mathbb{R}^{(m-1) \times (m-1)}.
\]
and $A_h \in \mathbb{R}^{(m-1) \times (m-1)}$ as before. Equivalent:
\[
\left(I - \frac{\tau}{2} M_h\right) w^{n+1} - \left(I + \frac{\tau}{2} M_h\right) w^n + \tau rK = 0.
\]
For vectors $(y_1, \ldots, y_d)^T$ and $(z_1, \ldots, z_d)^T$ with nonnegative entries, we have
\[
y_k \cdot z_k = 0 \text{ for all } k = 1, \ldots, d \iff y^T z = 0.
\]
This motivates the discretization
\[
\left(w^{n+1} - w^0\right)^T \left(\left(I - \frac{\tau}{2} M_h\right) w^{n+1} - \left(I + \frac{\tau}{2} M_h\right) w^n + \tau rK\right) = 0 \quad (8.6a)
\]
\[
\left(I - \frac{\tau}{2} M_h\right) w^{n+1} - \left(I + \frac{\tau}{2} M_h\right) w^n + \tau rK \geq 0 \quad (8.6b)
\]
\[
w^{n+1} - w^0 \geq 0 \quad (8.6c)
\]
for the linear complementary problem (8.5). This has to be solved for $n = 0, \ldots, N - 1$. 
8.3 An iterative method for linear complementary problems

Consider the linear complementary problem

\[(w - v)^T (Mw - b) = 0\]
\[Mw - b \geq 0\]
\[w - v \geq 0\]

with given \(M \in \mathbb{R}^{d \times d}\), given \(v, b \in \mathbb{R}^d\) and unknown \(w \in \mathbb{R}^d\). The problem (8.6) is obtained for

\[w = w^{n+1}, \quad v = w^0, \quad M := I - \frac{\tau}{2} M_h, \quad b = \left( I + \frac{\tau}{2} M_h \right) w^n - \tau r K .\]

Numerical method?

(a) Iterative methods for linear systems

First, consider only the linear system \(Mw = b\) with \(M = (M_{ij})_{ij} \in \mathbb{R}^{d \times d}\). Instead of direct methods (e.g. Gauss elimination) we consider iterative methods based on the decomposition

\[M = D - L - U\]

with

\[D = \begin{pmatrix} M_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_{dd} \end{pmatrix}, \quad L = - \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ M_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ M_{d1} & \cdots & M_{d,d-1} & 0 \end{pmatrix}, \quad U = - \begin{pmatrix} 0 & M_{12} & \cdots & M_{1d} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{d-1,d} \end{pmatrix}\]

Assume that \(D_{kk} = M_{kk} > 0\). This is the case, e.g., if \(M\) is symmetric and positive definite. By definition:

\[Mw = b \quad \iff \quad Dw = (L + U)w + b\]

Idea: Turn this into a fixed-point iteration. This yields the Jacobi iteration

\[Dw^{(j+1)} = (L + U)w^{(j)} + b, \quad j = 0, 1, 2, \ldots\]

Hope that the sequence \((w^{(j)})_{j \in \mathbb{N}_0}\) converges to a fixed-point.
Often a better convergence rate is observed with the **Gauss-Seidel iteration**:

\[(D - L)w^{(j+1)} = Uw^{(j)} + b, \quad j = 0, 1, 2, \ldots\]

In each step, a linear system with a triangular matrix has to be solved:

\[
w_1^{(j+1)} = \frac{1}{D_{11}} \left( 0 + \sum_{k=2}^{d} U_{1k}w_k^{(j)} + b_1 \right)
\]

\[
w_2^{(j+1)} = \frac{1}{D_{22}} \left( L_{21}w_1^{(j+1)} + \sum_{k=3}^{d} U_{2k}w_k^{(j)} + b_2 \right)
\]

\[
w_i^{(j+1)} = \frac{1}{D_{ii}} \left( \sum_{k=1}^{i-1} L_{ik}w_k^{(j+1)} + \sum_{k=i+1}^{d} U_{ik}w_k^{(j)} + b_i \right) \quad i = 1, \ldots, d.
\]

The Gauss-Seidel iteration is a special case of the **Successive Overrelaxation Method (SOR method)**

\[(D - rL)w^{(j+1)} = ((1 - r)D + rU)w^{(j)} + rb, \quad j = 0, 1, 2, \ldots\]

with relaxation parameter \( r \in \mathbb{R} \). If \( r = 1 \), then we recover the Gauss-Seidel method. The entries of the new iterate are computed by the following algorithm:

```plaintext
For j = 0, 1, 2, \ldots, w^{(j+1)} = w^{(j)} (will be overwritten)
    For k = 1, \ldots, d
        \[\tilde{w}_k^{(j)} = \frac{1}{D_{kk}} \left[ Lw^{(j+1)} + Uw^{(j)} + b \right]_k\] (k-th entry)
        \[w_k^{(j+1)} = w_k^{(j)} + r(\tilde{w}_k^{(j)} - w_k^{(j)})\]
    end
end
```

For \( r \in (0, 1) \), \( w^{(j+1)} \) is an interpolation between \( w^{(j)} \) and \( \tilde{w}^{(j)} \). In practice, however, typically \( r > 1 \) is chosen ("overrelaxation").

**(b) The projected SOR method for linear complementary problems**

Back to the linear complementary problem:

\[(w - v)^T (Mw - b) = 0 \quad \text{(8.7a)}\]

\[Mw - b \geq 0 \quad \text{(8.7b)}\]

\[w - v \geq 0 \quad \text{(8.7c)}\]
In general, we cannot expect that $Mw = b$. There are some entries $k$ where $[Mw - b]_k > 0$. The complementary problem is equivalent to

\[
\min \left\{ [Mw - b]_k, [w - v]_k \right\} = 0 \quad \text{for all } k = 1, \ldots, d
\]

\[
\iff \min \left\{ \left[ \frac{D^{-1}((D - L - U)w - b)}{w - D^{-1}(L+U)w + b} \right]_k, [w - v]_k \right\} = 0 \quad \text{for all } k = 1, \ldots, d
\]

\[
\iff \max \left\{ \left[ D^{-1}((L + U)w + b) \right]_k, v_k \right\} = w_k \quad \text{for all } k = 1, \ldots, d
\]

This motivates the **projected SOR method**:

\[
\text{For } j = 0, 1, 2, \ldots \]
\[
w^{(j+1)} = w^{(j)} \quad \text{(will be overwritten)}
\]

\[
\text{For } k = 1, \ldots, d
\]
\[
\hat{w}^{(j)}_k = \frac{1}{D_{kk}} \left[ Lw^{(j+1)} + Uw^{(j)} + b \right]_k
\]
\[
w^{(j+1)}_k = \max \left\{ w^{(j)}_k + r(\hat{w}^{(j)}_k - w^{(j)}_k), v_k \right\}
\]

end

end

**Theorem 8.3.1 (Cryer)** Let $v, b \in \mathbb{R}^d$, $r \in (1, 2)$ and assume that $M \in \mathbb{R}^{d \times d}$ is symmetric and positive definite. Then, the linear complementary problem (8.7) has a unique solution, and the iterates $w^{(j)}$ of the projected SOR method converge to the solution.

**Remark.** The theorem does not specify the order of convergence. Moreover, the matrix $M$ is in general not symmetric if $M = I - \frac{r}{2}M_h$. Hence, the theorem is only of limited use for our purpose.

### 8.4 Summary: Pricing American options with the projected SOR method

- Start: Free boundary problem (8.3) with solution $V(t, S)$.
- Reformulation as a linear complementary problem (8.4).
- Truncation, time inversion, define $u(t, S) = \tilde{V}(T - t, S) - f(S)$. Obtain new linear complementary problem (8.5).
- Discretize in time and space: $w^{n}_k \approx u(t_n, S_k)$.
- Algorithm:
For $n = 0, 1, \ldots, N - 1$ (time points)
    Solve linear complementary problem (8.6) with the
    projected SOR method
For $j = 0, 1, 2, \ldots$ (iteration number)
    For $k = 1, \ldots, m - 1$ (entry number)
        ...
    end
end

- $V(T - t_n, S_k) \approx w_k^n + f(S_k)$ for all $n$ and $k$.  

— FIN —
Appendix A

Some definitions from probability theory

Definition A.0.1 (Probability space) The triple \((\Omega, \mathcal{F}, \mathbb{P})\) is called a probability space, if the following holds:

1. \(\Omega \neq \emptyset\) is a set, and \(\mathcal{F}\) is a \(\sigma\)-algebra (or \(\sigma\)-field) on \(\Omega\), i.e. a family of subsets of \(\Omega\) with the following properties:
   - \(\emptyset \in \mathcal{F}\)
   - If \(F \in \mathcal{F}\), then \(\Omega \setminus F \in \mathcal{F}\)
   - If \(F_i \in \mathcal{F}\) for all \(i \in \mathbb{N}\), then \(\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}\)

The pair \((\Omega, \mathcal{F})\) is called a measurable space.

2. \(\mathbb{P} : \mathcal{F} \rightarrow [0, 1]\) is a probability measure, i.e.
   - \(\mathbb{P}(\emptyset) = 0\) and \(\mathbb{P}(\Omega) = 1\)
   - If \(F_i \in \mathcal{F}\) for all \(i \in \mathbb{N}\) are pairwise disjoint (i.e. \(F_i \cap F_j = \emptyset\) for \(i \neq j\)), then
     \[
     \mathbb{P}\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(F_i).
     \]

Definition A.0.2 (Borel \(\sigma\)-algebra) If \(\mathcal{U}\) is a family of subsets of \(\Omega\), then the \(\sigma\)-algebra generated by \(\mathcal{U}\) is

\[
\mathcal{F}_\mathcal{U} = \bigcap \left\{ \mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-algebra of } \Omega \text{ and } \mathcal{U} \subset \mathcal{F} \right\}.
\]

If \(\mathcal{U}\) is the collection of all open subsets of a topological space \(\Omega\) (e.g. \(\Omega = \mathbb{R}^d\)), then \(\mathcal{B} = \mathcal{F}_\mathcal{U}\) is called the Borel \(\sigma\)-algebra on \(\Omega\). The elements \(B \in \mathcal{B}\) are called Borel sets.

For the rest of this section \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space.
Definition A.0.3 (Measurable functions, random variables)

- A function \( X : \Omega \rightarrow \mathbb{R}^d \) is called \( \mathcal{F} \)-measurable if
  \[
  X^{-1}(B) := \{ \omega \in \Omega : X(\omega) \in B \} \in \mathcal{F}
  \]
  for all Borel sets \( B \in \mathcal{B} \). If \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, then every \( \mathcal{F} \)-measurable function is called a random variable.

- If \( X : \Omega \rightarrow \mathbb{R}^d \) is any function, then the \( \sigma \)-algebra generated by \( X \) is the smallest \( \sigma \)-algebra on \( \Omega \) containing all the sets \( X^{-1}(B) \) for all \( B \in \mathcal{B} \).

Notation: \( \mathcal{F}^X = \sigma\{X\} \)

\( \mathcal{F}^X \) is the smallest \( \sigma \)-algebra where \( X \) is measurable.

Definition A.0.4 (Independence)

- Two sets \( A \subset \Omega \) and \( B \subset \Omega \) are called independent if
  \[
  \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).
  \]

- For \( n \in \mathbb{N} \) let \( \mathcal{G}_1, \ldots, \mathcal{G}_n \subset \mathcal{F} \) be a collection of sub-\( \sigma \)-algebras of \( \mathcal{F} \). \( \mathcal{G}_1, \ldots, \mathcal{G}_n \) are independent if
  \[
  \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n) \quad \text{for all} \quad A_i \in \mathcal{G}_i.
  \]

- Random variables \( X_1, \ldots, X_n \) are called independent if
  \[
  \mathbb{P}\left( \bigcap_{i=1}^n X_i^{-1}(A_i) \right) = \prod_{i=1}^n \mathbb{P}(X_i^{-1}(A_i))
  \]
  for all \( A_1, \ldots, A_n \in \mathcal{B} \). Equivalent: The random variables \( X_1, \ldots, X_n \) are independent if the \( \sigma \)-algebras generated by \( X_1, \ldots, X_n \) are independent.

- Let \( X \) and \( Y \) be random variables with \( \mathbb{E}(|X Y|) \leq \infty \). Then \( X \) and \( Y \) are independent if and only if \( \mathbb{E}(X Y) = \mathbb{E}(X)\mathbb{E}(Y) \).

- A random variable \( X \) is independent of a sub-\( \sigma \)-algebra \( \mathcal{G} \subset \mathcal{F} \) if the \( \sigma \)-algebra generated by \( X \) is independent of \( \mathcal{G} \).
Appendix B

The Riemann-Stieltjes integral

Let \( f : [a, b] \to \mathbb{R} \) be bounded and let \( w : [a, b] \to \mathbb{R} \) be increasing, i.e. \( w(t) \geq w(s) \) for all \( t \geq s \). For a partition \( a = t_0 < t_1 < \ldots < t_N = b \) we define the lower and upper sums

\[
\overline{S}_N := \sum_{n=0}^{N-1} \sup \{ f(t) : t \in [t_n, t_{n+1}] \} (w(t_{n+1}) - w(t_n))
\]
\[
\underline{S}_N := \sum_{n=0}^{N-1} \inf \{ f(t) : t \in [t_n, t_{n+1}] \} (w(t_{n+1}) - w(t_n)).
\]

If \( \overline{S}_N \) and \( \underline{S}_N \) converge to the same value as the partition is refined, then the Riemann-Stieltjes integral is defined by

\[
\int_a^b f(t) \, dw(t) := \lim_{N \to \infty} \overline{S}_N = \lim_{N \to \infty} \underline{S}_N.
\]

For \( w(t) = t \), this is the standard Riemann integral.

If \( w : [a, b] \to \mathbb{R} \) is not increasing but has bounded variation, then there are increasing functions \( w_1 : [a, b] \to \mathbb{R} \) and \( w_2 : [a, b] \to \mathbb{R} \) such that \( w(t) = w_1(t) - w_2(t) \), and the Riemann-Stieltjes integral can be defined by

\[
\int_a^b f(t) \, dw(t) := \int_a^b f(t) \, dw_1(t) - \int_a^b f(t) \, dw_2(t).
\]
Appendix C

Runge-Kutta methods – a reminder

Consider the initial value problem
\[ y'(t) = f(t, y), \quad t \in [t_0, t_{\text{end}}], \quad y(0) = y_0 \]  
(C.1)

with appropriate function \( f : [t_0, t_{\text{end}}] \times \mathbb{R}^d \to \mathbb{R}^d \) (e.g. \( y \mapsto f(t, y) \) Lipschitz continuous). Choose \( N \in \mathbb{N} \), let \( \tau = (t_{\text{end}} - t_0)/N \) and \( t_n = t_0 + n\tau \).

Goal: Find approximations \( y_n \approx y(t_n), \ n = 0, 1, \ldots, N \). Ansatz:
\[
y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} y'(s) \, ds = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s)) \, ds
\]

Approximate the integral by the quadrature formula
\[
y(t_n + \tau) \approx y(t_n) + \tau \sum_{i=1}^{s} b_i f(t_n + c_i \tau, y(t_n + c_i \tau))
\]
with \( s \in \mathbb{N} \) and coefficients \( b_i \) and \( c_i \). Apply the same procedure to \( y(t_n + c_i \tau) \):

Approximate
\[
y(t_n + c_i \tau) = y(t_n) + \int_{t_n}^{t_n + c_i \tau} f(s, y(s)) \, ds
\]
\[
\approx y(t_n) + \tau \sum_{j=1}^{s} a_{ij} f(t_n + c_j \tau, y(t_n + c_j \tau)), \quad i = 1, \ldots, s
\]
with coefficients \( a_{ij} \). This yields the **Runge-Kutta method**: For each \( n = 0, \ldots, N - 1 \) solve the nonlinear system
\[
Y_i = y_n + \tau \sum_{j=1}^{s} a_{ij} f(t_n + c_j \tau, Y_j), \quad i = 1, \ldots, s \quad \text{(C.2a)}
\]
(e.g. by a version of Newton’s method) and let
\[
y_{n+1} = y_n + \tau \sum_{i=1}^{s} b_i f(t_n + c_i \tau, Y_i). \quad \text{(C.2b)}
\]
Each Runge-Kutta method is characterized by its coefficients $a_{ij}$, $b_i$, $c_j$. These are represented in the Butcher tableau:

\[
\begin{array}{c|ccc}
  c_1 & a_{11} & \cdots & a_{1s} \\
  \vdots & \vdots & & \vdots \\
  c_s & a_{s1} & \cdots & a_{ss} \\
  \hline
  b_1 & \cdots & b_s \\
\end{array}
\]

Examples:

- Explicit Euler method: $y_{n+1} = y_n + \tau f(t_n, y_n)$
- Implicit Euler method: $y_{n+1} = y_n + \tau f(t_{n+1}, y_{n+1})$
- Trapezoidal rule: $y_{n+1} = y_n + \tau \left( f(t_n, y_n) + f(t_{n+1}, y_{n+1}) \right)$
- Implicit midpoint rule: $y_{n+1} = y_n + \tau f(t_n + \frac{\tau}{2}, y_n + \frac{\tau}{2} y_{n+1})$

\[
\begin{array}{c|c}
  0 & 0 \\
  1 & 1 \\
  \hline
  \frac{1}{2} & \frac{1}{2} - c \\
  \frac{1}{2} + c & \frac{1}{4} & \frac{1}{4} - c \\
  \hline
  \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

explicit Euler implicit Euler trapezoidal rule impl. midp. rule

Gauss-Runge-Kutta method for $s = 2$

The Runge-Kutta method (C.2a), (C.2b) is explicit if $a_{ij} = 0$ for all $j \geq i$.

**Notation:** Let $\Psi_t(t_\ast, y_\ast)$ be the flow of the ODE $y'(t) = f(t, y)$, i.e. $y(t) = \Psi_{t-t_\ast}(t_\ast, y_\ast)$ is the exact solution of the initial value problem

$y'(t) = f(t, y), \quad t > t_\ast, \quad y(t_\ast) = y_\ast$.

The approximation of a Runge-Kutta method after $n$ steps with step-size $\tau$ and initial value $y_\ast$ at time $t_\ast$ is denoted by $\Phi^n_\tau(t_\ast, y_\ast)$; $\Phi_\tau$ is called the numerical flow.

**Definition C.0.1** A Runge-Kutta method applied to the ODE $y'(t) = f(t, y)$ with a sufficiently smooth function $f$ has order $p$ if

\[
\|\Psi_\tau(t_\ast, y_\ast) - \Phi_\tau(t_\ast, y_\ast)\| \leq C\tau^{p+1} \tag{C.3}
\]

for all $y_\ast \in \mathbb{R}^d$, $t_\ast \in \mathbb{R}$ and sufficiently small step-sizes $\tau > 0$. The constant may depend on $f$, $y_\ast$, and $t_\ast$ but must be independent of $\tau$. 
Remark: The term \( \| \Psi_t(t_*, y_*) - \Phi_t(t_*, y_*) \| \) is the local error, i.e. the approximation error after only one step of the method. If the method is stable, then (C.3) implies the bound

\[
\| y(t_n) - y_n \| \leq C r^p, \quad y_n = \Phi^p(t_0, y_0)
\]

for the global error after \( n \) steps; cf. chapter II.3 in [HNW10].

**Theorem C.0.2** A Runge-Kutta method with the property

\[
s \sum_{j=1}^s a_{ij} = c_i \quad \text{for all } i = 1, \ldots, s
\]

has order

- \( p = 1 \), if

\[
\sum_{i=1}^s b_i = 1,
\]

- \( p = 2 \), if in addition

\[
\sum_{i=1}^s b_i c_i = \frac{1}{2},
\]

- \( p = 3 \), if in addition

\[
\sum_{i=1}^s b_i c_i^2 = \frac{1}{3}, \quad s \sum_{i=1}^s s \sum_{j=1}^s b_i a_{ij} c_j = \frac{1}{6}.
\]

Proof. Taylor expansion of \( y(t_0 + h) \) and \( y_1 \) about \( h = 0 \) + long computations. See chapter XIV, Satz 76.6, in [HB09], or chapter II.2 in [HNW10].

Both Euler methods have order 1, the trapezoidal rule has order 2.
Appendix D

Sketch of the proof of the Itô formula (Theorem 2.6.2).

(i) Equation (2.11) is the shorthand notation for

\[ Y_t = Y_0 + \int_0^t \left( \partial_t F(s,X_s) + \partial_x F(s,X_s) \cdot f(s,X_s) + \frac{1}{2} \partial^2_{xx} F(s,X_s) \cdot g^2(s,X_s) \right) ds \]

\[ + \int_0^t \partial_x F(s,X_s) \cdot g(s,X_s) dW_s \]

Assume that \( F \) is twice continuously differentiable with bounded partial derivatives. (Otherwise \( F \) can be approximated by such functions with uniform convergence on compact subsets of \([0, \infty) \times \mathbb{R}\).) Assume that \((t,\omega) \mapsto f(t,X_t(\omega))\) and \((t,\omega) \mapsto g(t,X_t(\omega))\) are elementary functions. (Otherwise approximate by elementary functions.) Hence, there is a partition \(0 = t_0 < t_1 < \ldots < t_N = t\) such that

\[ f(t,X_t(\omega)) = f(0,X_0(\omega))1_{[0,t]}(t) + \sum_{n=1}^{N-1} f(t_n,X_{t_n}(\omega))1_{(t_n,t_{n+1})}(t) \]

and the same equation with \(f\) replaced by \(g\).

(ii) Notation: For the rest of the proof, we define

\[ f^{(n)} := f(t_n,X_{t_n}), \quad F^{(n)} := F(t_n,X_{t_n}), \]

\[ g^{(n)} := g(t_n,X_{t_n}), \quad \partial_t F^{(n)} := \partial_t F(t_n,X_{t_n}) \]

and so on, and

\[ \Delta t_n = t_{n+1} - t_n, \quad \Delta X_n = X_{t_{n+1}} - X_{t_n}, \quad \Delta W_n = W_{t_{n+1}} - W_{t_n}. \]
Since $f$ and $g$ are elementary functions, we have

\[
X_{t_n} = X_0 + \int_0^{t_2} f(s, X_s) \, ds + \int_0^{t_n} g(s, X_s) \, dW_s
\]

\[
= X_0 + \sum_{k=0}^{n-1} f(t_k, X_{t_k}) \Delta t_k + \sum_{k=0}^{n-1} g(t_k, X_{t_k}) \Delta W_k.
\]

and hence

\[
\Delta X_n = X_{t_{n+1}} - X_{t_n} = f^{(n)}(t_n) + g^{(n)}(t_n) \Delta W_n.
\]

(iii) Telescoping sum:

\[
Y_t = Y_{t_N} = Y_0 + \sum_{n=0}^{N-1} (Y_{t_{n+1}} - Y_{t_n}) = Y_0 + \sum_{n=0}^{N-1} \left( F^{(n+1)} - F^{(n)} \right)
\]

Apply Taylor’s theorem:

\[
F^{(n+1)} - F^{(n)} = \partial_t F^{(n)} \cdot \Delta t_n + \partial_x F^{(n)} \cdot \Delta X_n + \frac{1}{2} \partial_{tt} F^{(n)} \cdot (\Delta t_n)^2 + \partial_t \partial_x F^{(n)} \cdot \Delta t_n \Delta X_n
\]

\[
+ \frac{1}{2} \partial_{xx} F^{(n)} \cdot (\Delta X_n)^2 + R_n(\Delta t_n, \Delta X_n)
\]

with a remainder term $R_n$. Insert this into the telescoping sum.

(iv) Consider the limit $N \to \infty$, $\Delta t_n \to 0$ with respect to $\| \cdot \|_{L^2(dP)}$. For the first two terms, this yields

\[
\lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_t F^{(n)} \cdot \Delta t_n = \int_0^t \partial_t F(s, X_s) \, ds
\]

and

\[
\lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_x F^{(n)} \cdot \Delta X_n
\]

\[
= \int_0^t \partial_x F(s, X_s) \cdot f(s, X_s) \, ds + \int_0^t \partial_x F(s, X_s) \cdot g(s, X_s) \, dW_s.
\]
Next, we investigate the \( \partial_x^2 F^{(n)} \) term. Since
\[
(\Delta X_n)^2 = \left( f^{(n)} \Delta t_n + g^{(n)} \Delta W_n \right)^2
\]
we have
\[
\frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (\Delta X_n)^2 = \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (f^{(n)})^2 (\Delta t_n)^2 
+ \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot f^{(n)} g^{(n)} \Delta t_n \Delta W_n 
+ \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (g^{(n)})^2 (\Delta W_n)^2.
\]
\( \text{(D.1)} \) \( \text{(D.2)} \) \( \text{(D.3)} \)

For the right-hand side of \( \text{(D.1)} \), we obtain
\[
\left\| \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (f^{(n)})^2 (\Delta t_n)^2 \right\|_{L^2(dP)}^2 = \mathbb{E} \left[ \left( \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (f^{(n)})^2 (\Delta t_n)^2 \right)^2 \right] \rightarrow 0.
\]

With the abbreviation \( \alpha^{(n)} := \partial_x^2 F^{(n)} \cdot f^{(n)} g^{(n)} \) we obtain for the right-hand side of \( \text{(D.2)} \) that
\[
\left\| \sum_{n=0}^{N-1} \alpha^{(n)} \Delta t_n \Delta W_n \right\|_{L^2(dP)}^2 = \mathbb{E} \left[ \left( \sum_{n=0}^{N-1} \alpha^{(n)} \Delta t_n \Delta W_n \right)^2 \right] = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mathbb{E} \left( \alpha^{(n)} \alpha^{(m)} \Delta W_n \Delta W_m \right) \Delta t_n \Delta t_m.
\]

Since
\[
\mathbb{E} \left( \alpha^{(n)} \alpha^{(m)} \Delta W_n \Delta W_m \right) = \mathbb{E} \left( \alpha^{(n)} \alpha^{(m)} \Delta W_n \right) \mathbb{E} \left( \Delta W_m \right) = 0
\]
for \( n < m \) and similar for \( m < n \), only the terms with \( n = m \) have to be considered, which yields
\[
\left\| \sum_{n=0}^{N-1} \alpha^{(n)} \Delta t_n \Delta W_n \right\|_{L^2(dP)}^2 = \sum_{n=0}^{N-1} \mathbb{E} \left( (\alpha^{(n)})^2 (\Delta t_n)^2 \mathbb{E} \left[ (\Delta W_n)^2 \right] \right) \Delta t_n \Delta t_n \rightarrow 0.
\]

The third term \( \text{(D.3)} \), however, has a non-zero limit: We show that
\[
\lim_{N \to \infty} \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (g^{(n)})^2 (\Delta W_n)^2 = \frac{1}{2} \int_0^T \partial_x^2 F(s,X_s) \cdot \left( g(s,X_s) \right)^2 ds
\]
which yields the strange additional term in the Itô formula. With the abbreviation
\[ \beta^{(n)} = \frac{1}{2} \partial_{x}^{2} F^{(n)} \cdot (g^{(n)})^{2} \]
we have
\[
\left\| \sum_{n=0}^{N-1} \beta^{(n)} ((\Delta W_{n})^{2} - \Delta t_{n}) \right\|_{L^{2}(dP)}^{2} = E \left[ \sum_{n=0}^{N-1} \beta^{(n)} ((\Delta W_{n})^{2} - \Delta t_{n}) \right]^{2}
\]
\[
= E \left[ \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \beta^{(n)} \beta^{(m)} ((\Delta W_{n})^{2} - \Delta t_{n}) ((\Delta W_{m})^{2} - \Delta t_{m}) \right].
\]
For \( n < m \) we have
\[
E \left[ \beta^{(n)} \beta^{(m)} ((\Delta W_{n})^{2} - \Delta t_{n}) ((\Delta W_{m})^{2} - \Delta t_{m}) \right] = 0
\]
and vice versa for \( n > m \). Hence, only the terms with \( n = m \) have to be considered, and we obtain
\[
\left\| \sum_{n=0}^{N-1} \beta^{(n)} ((\Delta W_{n})^{2} - \Delta t_{n}) \right\|_{L^{2}(dP)}^{2} = E \left[ \sum_{n=0}^{N-1} (\beta^{(n)})^{2} ((\Delta W_{n})^{2} - \Delta t_{n})^{2} \right]
\]
\[
= \sum_{n=0}^{N-1} E \left[ (\beta^{(n)})^{2} \right] E \left[ ((\Delta W_{n})^{2} - \Delta t_{n})^{2} \right] \to 0
\]
because \( E \left[ ((\Delta W_{n})^{2} - \Delta t_{n})^{2} \right] = 2 \Delta t_{n}^{2} \) according to Exercise 4.

(vi) With essentially the same arguments, it can be shown that
\[
\lim_{N \to \infty} \frac{1}{2} \sum_{n=0}^{N-1} \partial_{t}^{2} F^{(n)} \cdot (\Delta t_{n})^{2} = 0
\]
\[
\lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_{t} \partial_{x} F^{(n)} \cdot \Delta t_{n} \Delta X_{n} = 0
\]
and that the remainder term from the Taylor expansion can be neglected when the limit is taken.
Bibliography


