Problem 5  (Bounds for American options)
Consider American options with strike price $K > 0$ and the same maturity time $T$ on an underlying with price $S(t) \geq 0$. Let $V_{C}^{Am}(t, S(t))$ and $V_{P}^{Am}(t, S(t))$ denote the values of an American call and put option, respectively. Show that the following bounds hold for all $t \in [0, T]$, again under the assumptions (A1)–(A5) from the lecture.

(a) $V_{C}^{Am}(t, S(t)) = V_{C}^{Eu}(t, S(t))$ 
(b) $e^{-r(T-t)} K \leq S(t) + V_{P}^{Am}(t, S(t)) - V_{C}^{Am}(t, S(t)) \leq K$
(c) $(K - S(t))^+ \leq V_{P}^{Am}(t, S(t)) \leq K$

Here, $V_{C}^{Eu}(t, S(t))$ denotes the value of a European call option on the same underlying with the same strike price $K$ and maturity time $T$.

**Hint:** Use the bounds for European options and Put-call parity.

**Reminder:** American options can be exercised at any time up to and including the maturity date $T$.

**Solution proposal:**

Note that the obvious inequalities $V_{C}^{Eu} \leq V_{C}^{Am}$ and $V_{P}^{Eu} \leq V_{P}^{Am}$ hold for American and European options. American options are at least as valuable as European options because they can be exercised at any time before and including the maturity date $T$.

(a) Suppose that $V_{C}^{Am}$ is exercised early at $t < T$ with the payoff $S(t) - K$, where $S(t) > K$. Now we use the bounds for European call options to obtain

$$V_{C}^{Am}(t, S(t)) \geq V_{C}^{Eu}(t, S(t)) \geq (S(t) - e^{-r(T-t)} K)^+ = S(t) - e^{-r(T-t)} K > S(t) - K.$$

Therefore, selling the option (for $V_{C}^{Am}(t, S(t))$) is more valuable than exercising the option at some point before maturity. Even though the European call cannot be exercised before $T$, it can be sold for at least $S(t) - K$. Consequently, we have that $V_{C}^{Am} = V_{C}^{Eu}$.

**Caution:** One could be tempted to use the same reasoning for a put option. The inequalities $V_{P}^{Am}(t, S(t)) \geq V_{P}^{Eu}(t, S(t)) \geq (e^{-r(T-t)} K - S(t))^+$ still hold, but the analogue of (1) does not.

(b) From Put-call parity, $V_{P}^{Eu} \leq V_{P}^{Am}$ and part (a), it follows that

$$e^{-r(T-t)} K = S(t) + V_{P}^{Eu} - V_{C}^{Eu} \leq S(t) + V_{P}^{Am} - V_{C}^{Am}.$$

We prove the second inequality by contradiction.

Suppose that $V_{C}^{Am}(t, S(t)) - V_{P}^{Am}(t, S(t)) - S(t) + K < 0$ for some $t \in [0, T]$. Let $T^{*} \leq T$ denote the exercise time of the American put option. We construct an arbitrage strategy.

Note that because $V_{C}^{Am}(t, S(t)) \geq (S(t) - e^{-r(T-t)} K)^+$ for all $t \in [0, T]$, we have $V_{C}^{Am}(T^{*}, S(T^{*})) \geq 0$ if $S(T^{*}) \leq K$, and $V_{C}^{Am}(T^{*}, S(T^{*})) \geq S(T^{*}) - K$ if $S(T^{*}) > K$. This explains the entries in the row for the call option in the following table.

<table>
<thead>
<tr>
<th>Action</th>
<th>Portfolio at $t$</th>
<th>Portfolio at $T^{*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell put</td>
<td>$-V_{P}^{Am}(t, S(t))$</td>
<td>$-(K - S(T^{*}))$</td>
</tr>
<tr>
<td>Buy call</td>
<td>$V_{C}^{Am}(t, S(t))$</td>
<td>$\geq 0$</td>
</tr>
<tr>
<td>Sell stock</td>
<td>$-S(t)$</td>
<td>$S(T^{*}) - K$</td>
</tr>
<tr>
<td>Invest K</td>
<td>$K$</td>
<td>$e^{r(T^{*}-t)} K$</td>
</tr>
<tr>
<td>Sum</td>
<td>$V_{C}^{Am} - V_{P}^{Am} - S + K &lt; 0$</td>
<td>$\geq 0$</td>
</tr>
</tbody>
</table>

(c) The inequality in (b) is equivalent to

$$e^{-r(T-t)} K - S(t) + V_{C}^{Am}(t, S(t)) \leq K - S(t) + V_{C}^{Am}(t, S(t)).$$

Using (a) and the bounds for a European call, we have

$$V_{P}^{Am}(t, S(t)) \leq K - S(t) + V_{C}^{Am}(t, S(t))$$

The other inequality is clear for $K \leq S(t)$. Let $K \geq S(t)$. If $V_{P}^{Am}(t, S(t)) < (K - S(t))^+$ was satisfied, then one could buy the put and immediately exercise it. This would yield an immediate risk-free gain of $K - S(t) - V_{P}^{Am}(t, S(t)) > 0$ and hence an arbitrage strategy.

**Comments on this problem:**

- Part (a) is only valid if no dividends on the underlying asset are paid.
- Inequality (b) is something like a “put-call-inequality” for American options.
Let us visualize the values of American and European options compared to the price of the underlying. We start with the call options. The dashed lines indicate the bounds given in Problem 4 and 5.

\[
V_{Eu}^C(t, S(t)) = V_{Am}^C(t, S(t)) = S(t) - (S(t) - Ke^{-r(T-t)})^+
\]

For the put options, we get the following image.

\[
V_{Eu}^P(t, S(t)) = V_{Am}^P(t, S(t)) = (Ke^{-r(T-t)} - S(t))^+
\]