Problem 9  (Martingales)
Show that each of the following processes is a continuous martingale with respect to the standard Brownian filtration. Here, \( \{W_t : t \in [0, T]\} \) is a standard Wiener process.

(a) \( W_t \)  
(b) \( W_t^2 - t \)  
(c) \( \exp(\alpha W_t - \frac{\alpha^2}{2} t) \)

Solution proposal:
Throughout this solution we use further properties of the conditional expectation:

1. If a random variable \( X \) is \( G \)-measurable, \( \mathbb{E}(X | G) = X \).
2. If a random variable \( X \) is independent of \( G \), then \( \mathbb{E}(X | G) = \mathbb{E}(X) \).
3. Factorization property: If \( Y \) is \( G \)-measurable, then \( \mathbb{E}(YX | G) = Y \mathbb{E}(X | G) \).

(a) We compute \( \mathbb{E}[W_t | \mathcal{F}_s] \) for \( 0 \leq s < t \):
   \[
   \mathbb{E}[W_t | \mathcal{F}_s] = \mathbb{E}[W_t - W_s + W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s | \mathcal{F}_s] + \mathbb{E}[W_s | \mathcal{F}_s] = W_s.
   \]
   Here we have used that \( W_t - W_s \) is independent of \( \mathcal{F}_s \) and that its mean is 0. The equation \( \mathbb{E}[W_s | \mathcal{F}_s] = W_s \) follows from the fact that \( W_s \) is \( \mathcal{F}_s \)-measurable.

(b) We first compute \( \mathbb{E}[W_t^2 | \mathcal{F}_s] \) for \( 0 \leq s < t \):
   \[
   \mathbb{E}[W_t^2 | \mathcal{F}_s] = \mathbb{E}[(W_t - W_s + W_s)^2 | \mathcal{F}_s] = \mathbb{E}[(W_t - W_s)^2 | \mathcal{F}_s] - 2 \mathbb{E}[(W_t - W_s)W_s | \mathcal{F}_s] + \mathbb{E}[W_s^2 | \mathcal{F}_s].
   \]
   We have (now using all of the three properties 1. – 3. from above)
   1. \( \mathbb{E}[(W_t - W_s)^2 | \mathcal{F}_s] = \mathbb{E}[(W_t - W_s)^2] = t - s \),
   2. \( \mathbb{E}[W_s^2 | \mathcal{F}_s] = W_s^2 \),
   3. \( \mathbb{E}[(W_t - W_s)W_s | \mathcal{F}_s] = W_s \mathbb{E}[W_t - W_s | \mathcal{F}_s] = 0 \),
   and hence \( \mathbb{E}[W_t^2 - t] = W_s^2 - s \).
   
   Remark: We have also shown that \( \mathbb{E}[W_t^2 | \mathcal{F}_s] = W_s^2 + t - s \). Therefore, \( W_t^2 \) is not a martingale with respect to the standard Brownian filtration.

(c) For \( s < t \), we start with
   \[
   \mathbb{E}[\alpha W_t - \frac{\alpha^2}{2} t | \mathcal{F}_s] = \mathbb{E}[\alpha W_t - \frac{\alpha^2}{2} t - e^{\frac{\alpha^2}{2} t} W_s | \mathcal{F}_s] = e^{\alpha W_s - \frac{\alpha^2}{2} t} \mathbb{E}[\alpha W_t - \frac{\alpha^2}{2} t | \mathcal{F}_s].
   \]
   Then we use the identity
   \[
   -\frac{(x - \alpha(t - s))^2}{2(t - s)} = -\frac{x^2}{2(t - s)} + \alpha x - \frac{\alpha^2(t - s)}{2}
   \]
   and \( W_t - W_s \sim \mathcal{N}(0, t - s) \) to compute
   \[
   \mathbb{E}[e^{\alpha(W_t - W_s)}] = \frac{1}{\sqrt{2\pi(t - s)}} \int_{-\infty}^{\infty} e^{\alpha x - \frac{x^2}{2(t - s)}} dx
   = e^{\frac{\alpha^2}{2}(t - s)} \cdot \frac{1}{\sqrt{2\pi(t - s)}} \int_{-\infty}^{\infty} e^{-\frac{(x - \alpha(t - s))^2}{2(t - s)}} dx
   = e^{\frac{\alpha^2}{2}(t - s)}.
   \]
   Hence,
   \[
   \mathbb{E}[e^{\alpha W_t - \frac{\alpha^2}{2} t} | \mathcal{F}_s] = e^{\alpha W_s - \frac{\alpha^2}{2} t} \mathbb{E}[e^{\alpha W_t - \frac{\alpha^2}{2} t} | \mathcal{F}_s] = e^{\alpha W_t - \frac{\alpha^2}{2} t} e^{\frac{\alpha^2}{2}(t - s)} = e^{\alpha W_t - \frac{\alpha^2}{2} s}.\]