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Numerical methods in mathematical finance

Winter semester 2018/19

Problem Sheet 1

Problem 1 (Arbitrage 1)

As always Mrs M starts her day reading the financial sector of her morning paper. She checks the current value of her favourite stock, which is currently at 50 £. Moreover, she learns that the value of call and put options with strike price 50 £ and maturity 1 year of her favourite stock are 5 £ and 4 £, respectively. Knowing that at the moment 45 £ invested in a bond are worth 50 £ after 1 year, Mrs M smiles and calls her bank advisor. Show that Mrs M does indeed have reasons to smile (supply a rigorous argument).

Problem 2 (Arbitrage 2)

Let $C_1(t)$ and $C_2(t)$ denote the values of two European call options at time t on the same underlying. Both calls have maturity $T = 1$ year but different strike prices $K_1 = 47.7$ € and $K_2 = 40$ €. At $t = 0$ the values of the two options are $C_1(0) = 5.2$ € and $C_2(0) = 12.4$ €. The risk-free interest rate is $r = 10\%$.
Devise an arbitrage strategy - if procurable.

Problem 3 (Butterfly Spread)

Combinations of options can be used for more effective speculation and hedging in realistic market environments. The following portfolio is called *Butterfly Spread*. It is constructed as follows:

- buy one European call option with strike price K_1
- buy one European call option with strike price K_3, where $K_3 > K_1$
- sell two European call options with strike price $K_2 = (K_1 + K_3)/2$

All options are on the same underlying with the same maturity date T.
Draw a sketch of the payoff diagram (in dependency of the price of the underlying) at time $t = T$.

Programming Exercise 1 (Order Reduction)

The purpose of this problem is to become familiar with the programming exercises in MATLAB. Since it is not connected to the content of the lecture, there is **no submission**.

We perform a numerical experiment concerning the approximation of ordinary differential equation and follow in the footsteps of Prothero and Robinson who proposed in 1974 to consider the differential equation

$$y'(t) = \lambda\big(y(t) - \phi(t)\big) + \phi'(t)\,, \quad y(t_0) = y_0 = \phi(t_0)\,, \quad \lambda < 0 \tag{1}$$

to study the accuracy of Runge Kutta methods for stiff differential equations.

In this exercise we want to approximate the solution of (1) using each of the following the numerical methods:

- Explicit Euler method
- Implicit Euler method
- Midpoint rule
- Gauß-RKV (s=2)

and investigate their behavior for different values of λ. Therefore, we consider (1) for $t \in [0, 10]$ and choose specifically the function $\phi(t) = \sin(t) + \cos(t/2)$.

Fortunately, we know that $y(t) = \phi(t)$ solves problem (1) such that we are able to compare our numerical approximations with the exact solution in this experiment.

Proceed as follows:

(i) Write a MATLAB script that computes approximations $y_n \approx y(t_n)$, $t_n = nh$, using each of the above numerical schemes for $\lambda = -1$ and various step-sizes h, e.g. $h = 10/2^k$ for $k = 3, \dots, 12$.

(ii) Compute the maximum error over time for each method and step-size and visualize your results in a convergence plot by plotting errors vs. step-size in a `loglog`-plot. Verify the expected order of convergence of each method.

(iii) Investigate the behavior of the methods for λ with higher absolute value by repeating the experiment for different λ. Choose values up to $\lambda = -10000$ (perhaps you want to omit the result of the explicit Euler method in the `loglog`-plot at some point).

The observed phenomenon is called *order reduction*. You may consult [1] for a detailed explanation.

References

[1] E. Hairer and G. Wanner. *Solving ordinary differential equations. II*, volume 14 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2010. Stiff and differential-algebraic problems, Second revised edition, paperback.

This sheet will be discussed in the problem class on **22nd October, 2018**

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.