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Numerical methods in mathematical finance

Winter semester 2018/19

Problem Sheet 2

Problem 4 (Bounds for European calls and puts)
In the lecture the inequality

$$(S(t) - \mathrm{e}^{-r(T-t)}K)^+ \le V_C(t, S(t)) \le S(t)$$

for all $t \in [0,T]$ and all $S(t) \ge 0$ was shown under the assumptions (A1)–(A5). Show the remaining inequality from Lemma 1.4.2, i.e.

$$(\mathrm{e}^{-r(T-t)}K - S(t))^+ \le V_P(t, S(t)) \le \mathrm{e}^{-r(T-t)}K$$

for all $t \in [0,T]$ and all $S(t) \ge 0$.

Problem 5 (Bounds for American options)
Consider American options with strike price $K > 0$ and the same maturity time T on an underlying with price $S(t) \ge 0$. Let $V_C^{\mathsf{Am}}(t, S(t))$ and $V_P^{\mathsf{Am}}(t, S(t))$ denote the values of an *American* call and put option, respectively. Show that the following bounds hold for all $t \in [0,T]$, again under the assumptions (A1)–(A5) from the lecture.

(a) $V_C^{\mathsf{Am}}(t, S(t)) = V_C^{\mathsf{Eu}}(t, S(t))$

(b) $e^{-r(T-t)}K \le S(t) + V_P^{\mathsf{Am}}(t, S(t)) - V_C^{\mathsf{Am}}(t, S_t) \le K$

(c) $(K - S(t))^+ \le V_P^{\mathsf{Am}}(t, S(t)) \le K$

Here, $V_C^{\mathsf{Eu}}(t, S(t))$ denotes the value of a *European* call option on the same underlying with the same strike price K and maturity time T.

Hint: Use the bounds for European options and Put-call parity.

Reminder: American options can be exercised at any time up to and including the maturity date T.

Problem 6 (Quadratic Variation of a Wiener process)
For given $T > 0$ let

$$P_N = (t_n)_{n=0}^N, \quad 0 = t_0 < t_1 < \cdots < t_N = T,$$

denote a partition of $[0,T]$ with $|P_N| = \max_n |t_n - t_{n-1}|$.

(a) Let $f\colon [0,T] \to \mathbb{R}$ be a function. The *quadratic variation* of f up to time T is defined as

$$QV_{0,T}(f) = \lim_{\substack{N\to\infty \\ |P_N|\to 0}} \sum_{n=1}^N \left(f(t_n) - f(t_{n-1})\right)^2.$$

Show that if f is continuously differentiable, then $QV_{0,T}(f) = 0$.

(b) Let $\{W_t \colon t \in [0,T]\}$ be a standard Wiener process. Show that

$$\lim_{\substack{N\to\infty \\ |P_N|\to 0}} \left\| \sum_{n=1}^N \left(W_{t_n} - W_{t_{n-1}}\right)^2 - T \right\|_{L^2(\mathrm{d}\mathbb{P})} = 0,$$

where $\|X\|_{L^2(\mathrm{d}\mathbb{P})} = \sqrt{\mathbb{E}[X^2]}$.

Hint: You may use without proof that $\mathbb{E}[X^4] = 3\sigma^4$ for $X \sim \mathcal{N}(0, \sigma^2)$.

Combine parts (a) and (b) to draw a conclusion for the Wiener process.

This sheet will be discussed in the problem class on **29th October, 2018**

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.