Supplementary material for Problem Sheet 2

Here we show that for a normal random variable $X \sim \mathcal{N}(0, \sigma^2)$, we have $\mathbb{E}[X^4] = 3\sigma^4$.

Proof. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

be the *probability density function* of X. We can compute the expectation of $g(X)$ via

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx,$$

at least for sufficiently smooth functions $g\colon \mathbb{R} \to \mathbb{R}$. (Some people call this the *law of the unconscious statistician*.)

Using integration by parts, we obtain

$$\begin{aligned}
\mathbb{E}[X^4] &= \int_{-\infty}^{\infty} x^4 \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \left(\frac{-x^3\sigma}{\sqrt{2\pi}}\right) \left(-\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \\
&= \left[-\frac{x^3\sigma}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_{-\infty}^{\infty} + 3\sigma^2 \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \\
&= 3\sigma^2 \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx. \qquad (1)
\end{aligned}$$

The term with square brackets vanishes because the function

$$-x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

tends to zero for $x \to \pm\infty$. The integral in (1) is just the variance of X and therefore equal to σ^2. Of course, one may also use integration by parts once again to obtain

$$\begin{aligned}
\mathbb{E}[X^4] &= 3\sigma^2 \int_{-\infty}^{\infty} \left(\frac{-x\sigma}{\sqrt{2\pi}}\right) \left(-\frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx \\
&= 3\sigma^2 \left(\left[-\frac{x\sigma}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx\right) \\
&= 3\sigma^4,
\end{aligned}$$

as the function

$$x \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

vanishes at $\pm\infty$, too. □

See the following picture (with $\sigma^2 = 1$) for the decay properties of the above mentioned functions.