Institute for Applied and Numerical Mathematics
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Numerical methods in mathematical finance

Problem Sheet 3

Winter semester 2018/19

Problem 7 (Itô's integral)
Let $\{W_t \colon t \in [0,T]\}$ be a standard Wiener process. Use the definition of Itô's integral for elementary functions to show that

$$\int_0^T W_t \mathrm{d}W_t = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

Hint: Show that $(\phi_k)_{k\in\mathbb{N}}$, defined as

$$\phi_k(t,\omega) = \sum_{n=0}^{k-1} W_{t_n}(\omega)\mathbf{1}_{(t_n,t_{n+1}]}(t) \quad \text{with} \quad t_n = \frac{nT}{k},$$

is the sequence of elementary functions from Lemma 2.3.4.
Then use the identity

$$W_{t_{n+1}}^2 - W_{t_n}^2 = (W_{t_{n+1}} - W_{t_n})^2 + 2W_{t_n}(W_{t_{n+1}} - W_{t_n})$$

and Problem 6(b).

Problem 8 (Conditional Expectation)
Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, X an integrable random variable and $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Show the following assertions.

(a) If X is a indicator random variable independent of $\mathcal{G}$, i.e. $X = \mathbf{1}_B$ where the set B is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

(b) If $\mathcal{G} = \{\emptyset,\Omega\}$, we have $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

(c) If $F \in \mathcal{F}$ with $\mathbb{P}(F) > 0$ and $\mathcal{G} = \{\emptyset, F, \Omega\setminus F, \Omega\}$, we have

$$\mathbb{E}(X|\mathcal{G})(\omega) = \begin{cases} \frac{1}{\mathbb{P}(F)}\int_F X\mathrm{d}\mathbb{P} & \text{if } \omega \in F, \\ \frac{1}{\mathbb{P}(\Omega\setminus F)}\int_{\Omega\setminus F} X\mathrm{d}\mathbb{P} & \text{if } \omega \in \Omega\setminus F. \end{cases}$$

Problem 9 (Martingales)
Show that each of the following processes is a continuous martingale with respect to the standard Brownian filtration. Here, $\{W_t \colon t\in[0,T]\}$ is a standard Wiener process.

(a) W_t (b) ${W_t}^2 - t$ (c) $\exp(\alpha W_t - \frac{\alpha^2}{2}t)$

Programming Exercise 2 (Euler-Maruyama method)
The price of the underlying asset is modeled by the Geometric Brownian Motion (GBM)

$$S(t) = S_0\exp(at + \sigma W(t)), \quad a = \mu - \sigma^2/2, \quad t \in [0,T]$$

with $\mu,\ \sigma > 0$, initial condition $S_0 \geq 0$ and $W(t)$ denoting the Wiener process. Furthermore, the process $S(t)$ is the solution of the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad t\in[0,T], \quad S(0) = S_0.$$

A numerical simulation of this SDE can be obtained by employing the *Euler-Maruyama method*: Choose $N \in \mathbb{N}$ and define the step-size $\tau = T/N$. For $n = 1,\dots,N$ let $t_n = n\tau$ and compute the approximation $S_n \approx S(t_n)$ as

$$S_n = S_{n-1} + \tau\mu S_{n-1} + \sigma S_{n-1}\Delta W_{n-1}$$

with $\Delta W_{n-1} = W(t_n) - W(t_{n-1})$.

In this problem we want to implement and test the *Euler-Maruyama method* in MATLAB. Proceed as follows:

(i) Write a MATLAB-function

```
[W, tspan] = wienerprocess(T,Nw,m)
```

which computes approximations of `m` paths of the Wiener process, each of them discretized with `Nw` points. Test your function with the parameters `T = 1`, `Nw =` 2^8, `m = 10` and plot the result.
Hint: Use the MATLAB-functions `randn` and `cumsum`.

(ii) Write a MATLAB-function

```
[Sapprox, tapprox] = eulerMaruyama(mu,sigma,S0,N,T,W)
```

which computes approximations of the GBM using the Euler-Maruyama method. `N` is the number of time-steps, `W` is a given discretized Wiener path.

(iii) Test your programs in a MATLAB-script `p2_main.m` as follows:

- Generate one fixed Wiener path with `T = 1` and `Nw =` 2^8.
- Compute the exact solution of the GBM for the Wiener path from above with `S0 = 1`, μ `= 0.3`, σ `= 0.4`.
- Use the Euler-Maruyama method to compute approximations with `N =` 2^6, 2^7, 2^8 using the given Wiener path.
  Caution: Choose the correct time points of the Wiener path!
- Plot the Euler-Maruyama approximations against time and compare them with the true solution of the GBM.

Please turn over.

Submission process:
In order to submit this exercise send an email containing your name and matriculation number to `benny.stein@kit.edu` and `uycyp@student.kit.edu`. The three MATLAB files (two functions, one script) should be included via an attachment. Add the subject line "Submission NumMethMathFin".

Check that you really added the attachments.

The submission deadline for this and all other programming exercises in the future is Sunday, February 17th.

The problems on this sheet will be discussed on **5th November, 2018**.

Assistance with this programming exercise will be provided in the tutorials on **6th November, 2018**, **7th November, 2018** and **8th November, 2018**.

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.