Numerical methods in mathematical finance

Winter semester 2018/19

Problem Sheet 4

Problem 10 (Itô's formula and analytical solution of SDEs)

Let $\{W_t \colon t \geq 0\}$ be a standard Brownian motion. Solve the stochastic differential equations

(a) $\mathrm{d}X_t = X_t\,\mathrm{d}t + \mathrm{d}W_t$,

(b) $\mathrm{d}X_t = \frac{1}{2}X_t\mathrm{d}t + X_t\mathrm{d}W_t$.

Hint: For (a) multiply with e^{-t}, and for (b) use the function $F(t,x) = \ln(x)$.

Problem 11 (Itô's formula)

Let the stochastic process $Y_t = 2 + t + \exp(W_t)$ be the solution of the stochastic differential equation

$$\mathrm{d}Y_t = a(t,Y_t)\,\mathrm{d}t + b(t,Y_t)\,\mathrm{d}W_t.$$

Using Itô's formula, determine the functions $a\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $b\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Problem 12 (Sketch of the proof of Itô's formula)

In this problem we discuss a proof of Itô's formula in the special case that the function F is time-independent (otherwise there are just more terms which have to be considered, but not more advanced techniques). We also assume that F is three times differentiable with bounded derivatives.

For simplicity, the functions $(t,\omega)\mapsto f(t,X_t(\omega))$ and $(t,\omega)\mapsto g(t,X_t(\omega))$ are stochastic step-functions, i.e.

$$f(t,X_t(\omega)) = f^{(0)}(\omega)\mathbf{1}_{[0,t_1]}(t) + \sum_{n=1}^{N-1} f^{(n)}(\omega)\mathbf{1}_{(t_n,t_{n+1}]}(t)$$

for a partition $0 = t_0 < t_1 < \ldots < t_N = t$ and the same equation with f replaced by g. In the general case one would approximate f and g by such functions.

The problem now is: Given a solution X_t of the SDE

$$\mathrm{d}X_t = f(t,X_t)\mathrm{d}t + g(t,X_t)\mathrm{d}W_t, \tag{1}$$

show that the equation

$$\mathrm{d}Y_t = \left(\partial_x F(X_t)f(t,X_t) + \frac{1}{2}\partial_x^2 F(X_t)g^2(t,X_t)\right)\mathrm{d}t + \partial_x F(X_t)g(t,X_t)\mathrm{d}W_t \tag{2}$$

holds for $Y_t := F(X_t)$.

Proceed as follows:

1.) Write down the integral equation for the process X_t to derive a simple formula for $\Delta X_n = X_{t_{n+1}} - X_{t_n}$ in terms of $f^{(n)}$ and $g^{(n)}$ and the increments $\Delta t_n = t_{n+1} - t_n$ and $\Delta W_n = W_{t_{n+1}} - W_{t_n}$.

2.) Apply Taylor's theorem to the function F and derive a formula for each of the differences $F^{(n+1)} - F^{(n)}$, where $F^{(n)} := F(X_{t_n})$. Then use a telescoping sum for $Y_t = Y_{t_N}$ in which you can insert the Taylor expressions.

3.) Now consider the limit $N\to\infty$, $\Delta t_n \to 0$ of the result of 2.) for Y_{t_N}. Start with the term without squares.

4.) Determine the limit of the second term (start with the binomial theorem, then examine each of the terms seperately). Use 1.) to replace ΔX_n by Δt_n and ΔW_n.

5.) Convince yourself that the remainder terms from Taylor's formula yield no additional contribution to Itô's formula.

The problems on this sheet will be discussed on **12th November, 2018**.

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.