Institute for Applied and Numerical Mathematics
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Numerical methods in mathematical finance

Winter semester 2018/19

Problem Sheet 7

Problem 17 (Strong and weak convergence)
Consider the SDE

$$\mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + g(t, X(t))\mathrm{d}W(t), \quad t \in [0, T], \quad X(0) = X_0$$

and the approximation

$$X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n)\Delta W_n\,,$$

where $N \in \mathbb{N}$, $\tau = T/N$ and $t_n = n\tau$.

Show that **strong** convergence of order γ implies **weak** convergence of order γ with respect to $F(x) = x$.

Problem 18 (Linear growth condition)
Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition

$$|f(x) - f(y)| \le L|x - y|$$

for all $x, y \in \mathbb{R}$.

Show that the function f satisfies a linear growth condition, too, i.e. there exists a constant $K \ge 0$ such that

$$|f(x)|^2 \le K(1 + |x|^2)$$

for all $x \in \mathbb{R}$.

Problem 19 (From the Binomial model to the Black-Scholes equation)
Consider a European call option with underlying $S(t)$ and maturity date $T > 0$ in the Binomial model, with the discretization of the interval $[0, T]$ given by

$$(t_n)_{n=0}^N = n\Delta t, \quad \Delta t = \tfrac{T}{N}$$

for $N \in \mathbb{N}$.

For all $n = 0, \dots, N$ and $j = 0, \dots, n$, denote by $S_{jn} = u^j d^{n-j} S(0)$ the price of the underlying at time t_n after j *up* and $n - j$ *down* price movements, and let V_{jn} denote the corresponding value of the option. Above, u and d are the factors by which the price of the underlying can go up or down in each interval, while p denotes the probability of an increase u.

Show that the approximation $V_{00} \approx V(0, S(0))$ is given by the *discrete* Black-Scholes formula

$$V_{00} = S(0)\Psi(m, N, q) - e^{-rT} K \Psi(m, N, p),$$

where $\Psi(m, N, p)$ is defined by the binomial distribution

$$\Psi(m, N, p) := \sum_{j=m}^{N} \binom{N}{j} p^j (1 - p)^{N-j}$$

with $m := \min\{0 \le j \le N \mid S_{jN} \ge K\}$ and $q = pue^{-r\Delta t}$.

The problems on this sheet will be discussed on **3rd December, 2018**.

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.