Institute for Applied and Numerical Mathematics
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Numerical methods in mathematical finance

Winter semester 2018/19

Problem Sheet 9

Problem 22 (Mean-square-error Monte-Carlo simulation)
In the lecture we observed how the Mean-square-error (MSE) of the Monte-Carlo simulation depends on the sample size m and the step-size τ of the discretization of the underlying SDE for each sample. It is thus reasonable to assume

$$\mathsf{MSE} = \xi(m,\tau) := \frac{c_1}{m} + c_2\tau^{2\gamma}$$

as an error model for the Monte-Carlo simulation with two constants c_1 and c_2.

(a) Argue why for some constant c_3,

$$C(m,\tau) := c_3\frac{m}{\tau}$$

is a reasonable model for the computational cost of the Monte-Carlo simulation.

(b) Compute the minimal error of the Monte-Carlo simulation for a given computational time C, i.e. minimize $\xi(m,\tau)$ with respect to m and τ subject to the side condition

$$c_3\frac{m}{\tau} = C$$

for a given C.

(c) Show that for the optimal m and τ from (b), one has

$$\sqrt{\mathsf{MSE}} = \sqrt{\xi(m,\tau)} = c_4 C^{-\frac{\gamma}{1+2\gamma}}\,.$$

Problem 23 (The Box-Muller transform)
Let $g\colon (0,1)^2 \to \mathbb{R}^2 \setminus \{0\}$ be given by

$$g(x) := \begin{pmatrix} \sqrt{-2\ln(x_1)}\cos(2\pi x_2) \\ \sqrt{-2\ln(x_1)}\sin(2\pi x_2) \end{pmatrix}, \quad x = (x_1, x_2) \in (0,1)\times(0,1).$$

(a) Show that its inverse $g^{-1}\colon g((0,1)^2) \to (0,1)^2$ is

$$g^{-1}(y) = \begin{pmatrix} \exp\left(-\frac{1}{2}(y_1^2+y_2^2)\right) \\ H(y_1,y_2) \end{pmatrix}$$

with

$$H(y_1,y_2) = \begin{cases} \frac{1}{2\pi}\arctan\left(\frac{y_2}{y_1}\right), & y_1>0,\ y_2>0, \\ \frac{1}{2\pi}\left(\arctan\left(\frac{y_2}{y_1}\right)+\pi\right), & y_1<0, \\ \frac{1}{2\pi}\left(\arctan\left(\frac{y_2}{y_1}\right)+2\pi\right), & y_1>0,\ y_2<0, \\ \frac{3}{4}, & y_1=0,\ y_2<0, \\ \frac{1}{4}, & y_1=0,\ y_2>0, \\ \frac{1}{2}, & y_2=0. \end{cases}$$

(b) Show that for the inverse g^{-1} from (a), we have

$$\left|\det J_{g^{-1}}(y)\right| = \frac{1}{2\pi}\exp\left(-\frac{1}{2}(y_1^2+y_2^2)\right) \quad \text{for} \quad y_1 \neq 0,$$

with $J_{g^{-1}}$ denoting the Jacobian of g^{-1}.

(c) Illustrate the transformation g via a suitable sketch.

The problems on this sheet will be discussed on **17th December, 2018**.

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.