Problem 25 (Discrepancy)
Let $R \subseteq [0,1]^d$ be an arbitrary axially parallel $d$-dimensional rectangle in the unit cube $[0,1]^d$. The discrepancy of the point set $P = \{x^1,\ldots,x^m\}$ is
\[
D_m := \sup_R \left| \frac{A(P,R)}{m} - \vol(R) \right|, \quad A(P,R) = \#(P \cap R),
\]
with the star discrepancy $D^*_m$ defined in the same way but maximizing the difference over those rectangles $R^*$ which are anchored in the origin, i.e., $R^* = \times_{i=1}^d [0, y_i)$ for some $y \in \mathbb{R}^d$.

(a) Show that the discrepancies $D_m$ and $D^*_m$ satisfy
\[
D^*_m \leq D_m \leq 2^d D^*_m.
\]

(b) Show that we have the lower bounds
\[
D_m \geq \frac{1}{m} \quad \text{and} \quad D^*_m \geq \frac{1}{2^d m}.
\]

(c) Let $d = 1$. Show that the sequence
\[
x^j = \frac{2j - 1}{2m} \quad \text{for} \quad j = 1,\ldots,m
\]
has $D^*_m = 1/(2m)$. Compare this with the second estimate in part (b).

Problem 26 (Difference quotients)
Let $y \in C^4([a,b])$. Choose $1 < m \in \mathbb{N}$, let $h = (b-a)/m$ and $x_k = a + kh$ for $k = 0,\ldots,m$. Show that the second derivative of $y$ can be approximated as follows:
\[
\max_{k=1,\ldots,m-1} \left| y''(x_k) - \frac{y(x_k + h) - 2y(x_k) + y(x_k - h)}{h^2} \right| \leq \frac{h^2}{12} \left\| \frac{d^4 y}{dx^4} \right\|_{\infty}
\]