Institute for Applied and Numerical Mathematics
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Numerical methods in mathematical finance

Problem Sheet 12

Winter semester 2018/19

Problem 27 (Spatial discretization of the heat equation)
Consider the heat equation on an interval $[a, b]$ with inhomogeneous Dirichlet boundary conditions given by functions $u_a(t)$ and $u_b(t)$. In the lecture, it was shown that the spatial discretization w_h via finite differences satisfies the ODE

$$w_h'(t) = A_h w_h(t) + g_h(t), \qquad t \in (0, t_{\text{end}}], \tag{1a}$$

$$w_h(0) = \overline{u}_0 \tag{1b}$$

with the finite difference matrix $A_h \in \mathbb{R}^{(m-1)\times(m-1)}$,

$$g_h(t) = \frac{1}{h^2}(u_a(t), 0, \dots, 0, u_b(t))^\top \in \mathbb{R}^{m-1}$$

and some $\overline{u}_0 \in \mathbb{R}^{m-1}$.

(a) Let

$$\widetilde{u}(t, x) = u_a(t) + \frac{u_b(t) - u_a(t)}{b - a}(x - a)$$

be the linear interpolant between $u_a(t)$ and $u_b(t)$ and set

$$\widetilde{w}(t) = (\widetilde{u}(t, x_1), \dots, \widetilde{u}(t, x_{m-1}))^\top.$$

Show that $\widehat{w}_h = w_h - \widetilde{w}$ is the solution of

$$\widehat{w}_h'(t) = A_h \widehat{w}_h(t) + \overline{f}(t), \qquad t \in (0, t_{\text{end}}], \tag{2a}$$

$$\widehat{w}_h(0) = w_h(0) - \widetilde{w}(0) \tag{2b}$$

with $\overline{f}(t) = -\partial_t \widetilde{w}(t)$.

(b) Compare the structure of problem (2) to the structure of the continuous problem (7.6) in the lecture notes.

Programming Exercise 6 (Finite differences for the Poisson problem)
In this programming exercise, we implement a finite difference scheme for a stationary diffusion problem on an interval $[-L, L]$. Here, we consider the equation

$$-\partial_x^2 u(x) = f(x), \qquad x \in (-L, L),$$

$$u(-L) = u(L) = 0$$

for a given function $f \colon [-L, L] \to \mathbb{R}$.

(a) Use equidistant finite differences with $m + 1$ points in space ($m - 1$ inner points) as in the lecture to derive a corresponding discrete problem $-A_h u_h = \overline{f}$ with $A_h \in \mathbb{R}^{(m-1)\times(m-1)}$ and $\overline{f} \in \mathbb{R}^{m-1}$.

(b) Write a MATLAB script `p6_main.m` in which you approximate the solution of the above equation for the function

$$f(x) = \frac{\pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right),$$

for which you know the exact solution u (which one is it?). Compute the error of your approximations in the scaled norm $|\cdot|_h$ from the lecture (or the maximum norm $|\cdot|_\infty$) for different discretization parameters m and plot the result (with a double logarithmic scale). You should observe second order convergence.

(c) Now test your script with the function

$$f(x) = -6|x|,$$

for which the exact solution is $u(x) = \text{sign}(x)x^3 - L^3$.

What do you observe here? Did you expect this behaviour?

The problems on this sheet will be discussed on **21st January, 2019**.

The link http://www.math.kit.edu/ianm3/edu/nummethmathfin2018w/en/ leads to the web page of the lecture. Here you will find all up-to-date information about the lecture and the problem class.