Karlsruher Institut für Technologie

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Aspects of Numerical Time Integration — Exercise Sheet 01 May 2, 2017

The aim of this exercise sheet is to practice the implementation of simple time integration schemes in MATLAB such as the explicit Euler method.

For $T > 0$ we consider the ordinary differential equation (ODE)

$$y'(t) = f(y(t)), \quad y(0) = y_0 \in \mathbb{R}^d, \qquad f : \mathbb{R}^d \to \mathbb{R}^d \text{ smooth}, \qquad t \in [0, T] \tag{1}$$

and discretize the time interval $t \in [0, T]$ such that $t_n = n\tau,\ n = 0, 1, 2, \ldots, N, t_N = T$, with a small **time step size** $\tau \in (0, 1)$.

A numerical time integration scheme for (1) should satisfy $y_n \approx y(t_n),\ \forall t_n \leq T$, i.e. the numerical method should approximate the exact solution of the ODE at every time $t_n \in [0, T]$.

The **flow** φ_f of a differential equation is defined as the mapping of an initial value y_0 to the exact solution $y(t)$ at some time $t \in [0, T]$. Then $\varphi_f^t(y_0) := y(t)$.

We say a numerical method $\Phi^\tau(y_n) := y_{n+1}$ is **consistent** of/ has a **local error** of order p if the local error satisfies

$$\left\| \Phi^\tau\Big(y(t_n)\Big) - \varphi_f^\tau\Big(y(t_n)\Big) \right\| \leq C\tau^{p+1}.$$

Exercise 1:

a) Show that the explicit Euler method

$$y_{n+1} = y_n + hf(y_n)$$

applied to (1) is consistent of order $p = 1$.

b) Consider the first order ODE

$$y'(t) = i\,|y(t)|^2 y(t), \qquad y(0) = y_0.$$

- Can you give the **exact solution** $y(t)$ explicitly?

Implement the explicit Euler method applied to this ODE with $y_0 = \frac{1-2i}{\sqrt{5}}$ in MATLAB with time step size $\tau = 0.1$ on the time interval $t \in [0, T = 10]$ and compare the numerical solution with the exact solution, i.e. plot the evolution of

$$|y_n - y(t_n)| \qquad \text{over all times } t_n.$$

- What can you see?
- Can you give an explanation for your observation?

c) Create an order plot (explanation below) and show that we find the order of consistency $p = 1$ also numerically. Thus, for various time step sizes $\tau_j = 2^{-j}, j \in \{3, 4, \ldots, 10\}$ run the simulation with the explicit Euler method and `loglog`-plot the maximal error

$$\text{err}_j = \max_{t_{n_j} \in [0,T]} |y_{n_j} - y(t_{n_j})|, \qquad \text{for each } \tau_j$$

against the corresponding time step size τ_j.

You should see a line with slope 1.

- Why do we see a simple line with a specific slope?
- Why don't we observe a line with slope 2?

Discussion in the problem class thursday 3:45 pm, in room 3.061 in the Kollegiengebäude Mathematik 20.30.

♣ : Please try to do exercises marked with ♣ at home.