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Aspects of Numerical Time Integration — Exercise Sheet 02 May 19, 2017

On this exercise sheet we recall the definition and the practical implementation of splitting methods.

For $T > 0$ and $t \in [0, T]$ we consider the ODE

$$y'(t) = f^{[1]}(y(t)) + f^{[2]}(y(t)) := f(y(t)), \quad y(0) = y_0 \in \mathbb{R}^d, \quad f^{[1]}, f^{[2]} \text{ smooth} \tag{1}$$

and denote its exact flow by $\varphi_f^t(y_0)$.

We split up the right hand side of (1) into the subproblems

$$v'(t) = f^{[1]}(v(t)), \quad v(0) = v_0 \in \mathbb{R}^d, \tag{S1}$$

$$w'(t) = f^{[2]}(w(t)), \quad w(0) = w_0 \in \mathbb{R}^d. \tag{S2}$$

with exact flows $\varphi^t_{f^{[1]}}(v_0), \varphi^t_{f^{[2]}}(w_0)$.

Under the assumption that these subproblems are easier to solve than the original problem (1) or even exactly solvable, splitting methods allow us to efficiently compute stable and accurate numerical approximations with time step sizes $0 < \tau < 1$.

In order to write down the approximation property of the local error $\left\| \Phi^\tau(y_0) - \varphi^\tau_f(y_0) \right\|_X \leq C\tau^{p+1}$ for a method $\Phi^\tau(y_0)$ of order p to the exact solution $\varphi^\tau_f(y_0)$, it is convenient to use the more simple $\mathcal{O}$-notation in the sense of the $\|\cdot\|_X$ norm, i.e.

$$\Phi^\tau(y_0) = \varphi^\tau_f(y_0) + \mathcal{O}\left(\tau^{p+1}\right) \quad \text{means that} \quad \left\| \Phi^\tau(y_0) - \varphi^\tau_f(y_0) \right\|_X \leq C\tau^{p+1},$$

where the constant C can be chosen independently of τ.

Then, we obtain for example with the **Lie splitting method**

$$\Phi^\tau_L(y_0) := \varphi^\tau_{f^{[2]}} \circ \varphi^\tau_{f^{[1]}}(y_0) = \varphi^\tau_f(y_0) + \mathcal{O}(\tau^2),$$

and with the **Strang splitting method**

$$\Phi^\tau_S(y_0) := \varphi^{\tau/2}_{f^{[1]}} \circ \varphi^\tau_{f^{[2]}} \circ \varphi^{\tau/2}_{f^{[1]}}(y_0) = \varphi^\tau_f(y_0) + \mathcal{O}(\tau^3).$$

The constants in the $\mathcal{O}$-terms only depend on the smoothness of $f^{[j]}$, $j = 1, 2$ and on the norm of the initial data.

Exercise 2:

Lie splitting versus Explicit Euler:

a) ♣ Let $0 < \tau < 1$. For the example of the linear case, i.e. $f(y) = Ay + By$, $A, B \in \mathbb{R}^{d\times d}$, show that the Lie splitting method Φ^τ_L is consistent of order $p = 1$ (cf. Exercise Sheet 1).

b) Consider the ODE

$$y'(t) = i\Lambda y(t) + i|y(t)|^2 y(t), \quad y(0) = y_0 \in \mathbb{R}^2, \quad t \in [0, T], \qquad \Lambda \in \mathbb{R}^{2\times 2}. \tag{2}$$

• Explain why a splitting method makes sense for this problem.

We set $y_0 = \frac{\tilde{y}_0}{\|\tilde{y}_0\|}$ with $\tilde{y}_0 = \begin{pmatrix} 1-2i \\ 3-4i \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}$ with $\omega \in \mathbb{R}$. Choose $\omega = 2, 10, 100$.

• In MATLAB first implement the **Lie splitting method** applied to (2) with time step size $\tau = 2^{-5}$ on the time interval $t \in [0, T = 10]$. Plot the modulus of the numerical solution, i.e. plot $|y_n|^2$ over all times $t_n < T$. Then do the same for the **explicit Euler method** in the same program.

Please turn over →

Reference solutions:

For many problems, in particular for (2), it is not easy and very often also impossible to write down an explicit formula for the exact solution.

But if we want to investigate the error of a numerical method Φ^τ applied to such a problem numerically we need so called numerical **reference solutions**.

Of course a reference solution should be of better numerical quality than the solution obtained with the method Φ^τ. Hence in order to compute a reference solution we choose a numerical method which is of a higher order of consistency than Φ^τ and use a smaller time step size $\tau_R = \tau/M$.

c) In MATLAB implement the **Strang splitting method** $\Phi_S^{\tau_R}$ applied to (2) as a reference method for the methods of part b) on the interval $t \in [0, T = 1]$ with time step size $\tau_R = \tau/8$.

• Why is the Strang splitting method a suitable choice for a reference method?

• As on exercise sheet 1 plot the evolution of the error of the Lie splitting method and the explicit Euler method and create an order plot for these methods. What can you observe?

Discussion in the problem class thursday 3:45 pm, in room 3.061 in the Kollegiengebäude Mathematik 20.30.

♣ : Please try to do exercises marked with ♣ at home.