Karlsruher Institut für Technologie

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Aspects of Numerical Time Integration — Exercise Sheet 03 June 16, 2017

Consider the partial differential equation (PDE)

$$\partial_{tt}u(t,x) = \partial_{xx}u(t,x) - u(t,x), \quad u(0,x) = \sin(x),\ \partial_t u(0,x) = -\sqrt{2}\cos(x), \quad t \in [0,T], \quad x \in [-\pi,\pi], \tag{1a}$$

equipped with periodic boundary conditions, i.e.

$$u(t,-\pi) = u(t,\pi), \quad \partial_x u(t,-\pi) = \partial_x u(t,\pi). \tag{1b}$$

The aim of this exercise sheet is to implement the "Störmer-Verlet" (SV) time integration scheme combined with a finite difference (FD) scheme applied to the *linear wave equation* (1).

Exercise 3:

♣ Show that $\tilde{u}(t,x) = \sin(x - \sqrt{2}t)$ is the exact solution of (1).

Whenever we intend to numerically solve a PDE involving time derivatives as well as spatial derivatives, we have to choose besides the time integration scheme also a suitable space discretization scheme.

Very popular is the **finite differences** space discretization scheme which we explain in the following.

Let $a,b \in \mathbb{R}$. Let $g : [a,b] \to \mathbb{C}$, $g \in C^4((a,b))$ be a smooth $[a,b]$-periodic function, i.e. $g(a) = g(b)$. Furthermore let $N_x \in \mathbb{N}$ and

$$x_j = x_0 + jh,\ j = 1,\dots,N_x \quad \text{with} \quad x_0 = a,\ x_{N_x} = b \quad \text{and} \quad h = (b-a)/N_x,$$

be a discretization of the interval $[a,b]$. In particular, we have that $g(x_0) = g(x_{N_x})$. In the notation $g^j = g(x_j), j = 0,\dots,N_x-1$ we thus have the following approximation property, i.e.

$$\partial_{xx}g(x_j) \approx \frac{1}{h^2}(g^{j-1} - 2g^j + g^{j+1}) =: g^j_{xx}. \tag{2}$$

Exercise 4:

a) ♣ Show that the approximation error of the finite difference scheme (2) satisfies

$$|\partial_{xx}g(x_j) - g^j_{xx}| \le Ch^2, \quad \text{for sufficiently smooth } g.$$

b) Find a matrix $A \in \mathbb{R}^{N_x \times N_x}$ such that

$$\partial_{xx}g(x_j) \approx (A\tilde{g})_j, \quad j = 0,\dots,N_x-1,$$

where $\tilde{g} = [g^0, g^1, \dots, g^{N_x-1}]^T$. Mind the periodic boundary conditions $g(a) = g(b)$.

please turn over →

Applying the finite difference space discretization scheme (2) to the PDE (1), the latter wave equation reduces to a discretized problem which is only time dependent, i.e.

$$\tilde{u}''(t) = A\tilde{u}(t) - \tilde{u}(t), \quad \tilde{u}(0) = (\sin(x_j))_{j=0}^{N_x-1},\ \tilde{u}'(0) = (-\sqrt{2}\cos(x_j))_{j=0}^{N_x-1}, \quad t \in [0,T], \tag{3}$$

where $(\tilde{u}(t))^j \approx u(t, x_j)$. Note that $\tilde{u}(t) = [(\tilde{u}(t))^0, \ldots, (\tilde{u}(t))^{N_x-1}]^T \in \mathbb{R}^{N_x}$ is a vector in $\mathbb{R}^{N_x}$.

Next we apply the *Störmer-Verlet* time integration scheme to the discretized problem (3) which we explain in the following.

Let $0 < \tau < 1$ be a small time step size and let $t_n = n\tau, n = 0, 1, 2, \ldots, N,\ t_N \leq T$ be a discretization of the time interval $[0, T]$. Furthermore let $u_n \approx \tilde{u}(t_n), v_n \approx \tilde{u}'(t_n)$ approximate the exact solution of (3).

Then the *Störmer-Verlet* time integration scheme applied to (3) is given by

$$\begin{aligned} v_{n+\frac{1}{2}} &= v_n + \frac{1}{2}\tau\,(Au_n - u_n) \\ u_{n+1} &= u_n + \tau v_{n+\frac{1}{2}} \\ v_{n+1} &= v_{n+1/2} + \frac{1}{2}\tau\,(Au_{n+1} - u_{n+1})\,. \end{aligned} \tag{SV}$$

Exercise 5:

a) In MATLAB implement the **Störmer-Verlet time integration method** applied to (1) on the spatial interval $x \in [-\pi, \pi]$ and on the time interval $t \in [0, T = 1]$ with time step size $\tau = 2^{-8}$ (cf. (SV)) using **finite differences** for the spatial discretization (cf. (2)) with $N_x = 8$ grid points and hence with spatial step size $h = (b - a)/N_x$.

b) Compare your numerical solution with the exact solution (see Exercise 3) for various number of grid points $N_x^{(\ell)} = 8 \cdot 2^\ell, \ell = 0, \ldots, 7$ at a **fixed** time step size $\tau = 2^{-8}$.

   Create an order plot, which shall allow you to see the order m of the spatial error, i.e. if the spatial error is of order $\mathcal{O}(h^k)$ you should observe a line with slope k.

   Do you have an explanation for the behaviour of the line at $N_x^{(7)} = 2^{10}$?

c) Compare your numerical solution with the exact solution for various time step sizes $\tau^{(m)} = 2^{-10} \cdot 2^m, m = 0, \ldots, 6$ at a **fixed** number of grid points $N_x = 150$ and create an order plot to see the order p of the time integration error, i.e. you should see a line with slope p if the global time integration error is of order $\mathcal{O}(\tau^p)$.

   Do you have an explanation for the behaviour of the line for small values of τ?

   Now change the number of grid points to $N_x = 256$. How does the line change and how can you explain this?

Discussion in the problem class thursday 3:45 pm, in room 3.061 in the Kollegiengebäude Mathematik 20.30.

♣ : Please try to do exercises marked with ♣ at home.