Aspects of Numerical Time Integration — Exercise Sheet 05

On this exercise sheet our aim is to compare the spatial order of finite differences and spectral methods applied to a time dependent problem. In particular we discuss the efficiency of both methods. We consider the nonlinear Schrödinger equation (NLS) on \((t, x) \in [0, T] \times \mathbb{K}\), i.e.

\[
1 \frac{\partial u}{\partial t} = i \Delta u + |u|^2 u, \quad u(0, x) = u_0(x), \quad x \in \mathbb{K},
\]

where at first we set \(\mathbb{K} = \mathbb{R}\) the real line.

**Exercise 8:**
Show that in this setting the function

\[
\psi_\alpha(t, x) = \frac{\sqrt{2\alpha}}{\cosh(x\sqrt{\alpha})} e^{i\alpha t}, \quad \alpha \in \mathbb{R}
\]

solves (NLS) with initial value \(u_0(x) = \psi_\alpha(0, x)\).

**Hint:** \(\cosh(x) = (e^x + e^{-x})/2, \quad \frac{\partial}{\partial x} \cosh(x) = \sinh(x), \quad \cosh(x)^2 = 1 + \sinh(x)^2\).

For practical implementation issues we introduce periodic boundary conditions for the solution of (NLS), i.e. we restrict ourselves to the torus \(T_L := [-L\pi, L\pi]\) for some \(L > 0\). More precisely, we replace \(\mathbb{K} = T_L\) in (NLS).

Now we choose \(L\) large enough, such that the boundary conditions are neglectable in the numerical solution.

**Exercise 9:**
Set \(T = 1, L = 2, \alpha = 8\) and choose a time step size \(\tau = 2^{-11}\). Furthermore set \(u_0(x) = \psi_\alpha(0, x)\). Choose \(N = 32\) grid points for spatial discretization.

a) In MATLAB implement the Strang splitting method applied to (NLS) for the space discretization with spectral methods and for the space discretization with finite differences.

b) Run both methods with \(N_1 = 32, N_2 = 64, N_3 = 128\) and \(N_4 = 256\) grid points and measure the elapsed time of each method for \(N = N_\ell, \ell = 1, 2, 3, 4\), using the MATLAB `tic, toc` commands. What can you observe? Which method is “faster” and why?

c) Create an order plot for the spatial order of both methods using the values of \(N\) from above by comparing the numerical solution with the exact solution \(\psi_\alpha(t, x)\) in the approximate \(L^2\)-norm, i.e.

\[
err_{N_\ell} = \sqrt{N_\ell} \max_{t \in [0, T]} \left\| u^{num,N_\ell}(t) - \psi_\alpha^{N_\ell}(t) \right\|_{L^2}, \quad \ell = 1, 2, 3, 4,
\]

where \(u^{num,N_\ell}(t)\) is an array with approximation to \(u(t, \cdot)\) in the grid points corresponding to \(N_\ell\), and where \(\psi_\alpha^{N_\ell}(t)\) is an array with the values of the exact solution \(\psi_\alpha(t, \cdot)\) in the grid points corresponding to \(N_\ell\), i.e.

\[
u_{num,N_\ell}(t) = \left( u^{num,N_\ell}(t, x_j) \right)_{j=1}^{N_\ell} \quad \text{and} \quad \psi_\alpha^{N_\ell}(t) = \left( \psi_\alpha^{N_\ell}(t, x_j) \right)_{j=1}^{N_\ell}.
\]

How does your plot change if you use a time step size \(\tau = 2^{-12}\), \(\tau = 2^{-13}\) or \(\tau = 2^{-14}\) instead? Can you give an explanation?

d) Now reset \(\tau = 2^{-11}\).

Create an efficiency plot for both methods, i.e. create a \(\log_{10}\) plot showing the time which is needed to obtain a specific error.

Therefore on the X-axis insert the error, which each method produces at the number of grid points \(N_\ell, \ell = 1, 2, 3, 4\) and on the Y-axis put the corresponding elapsed time.

Which of the schemes shows a better efficiency?

Discussion in the problem class Thursday 3:45 pm, in room 3.061 in the Kollegiengebäude Mathematik 20.30.