Karlsruher Institut für Technologie

Department of Mathematics
Institute for Applied and
Numerical Mathematics

JProf. Dr. Katharina Schratz
Dipl-Math. techn. Patrick Krämer

Aspects of Numerical Time Integration — Exercise Sheet 05 July 20, 2017

On this exercise sheet our aim is to compare the spatial order of **finite differences** and **spectral methods** applied to a time dependent problem. In particular we discuss the efficiency of both methods.

We consider the nonlinear Schrödinger equation (NLS) on $(t, x) \in [0, T] \times \mathbb{K}$, i.e.

$$1 \cdot \partial_t u = i\Delta u + i|u|^2 u, \quad u(0, x) = u_0(x), \quad x \in \mathbb{K}, \tag{NLS}$$

where at first we set $\mathbb{K} = \mathbb{R}$ the real line.

Exercise 8:

Show that in this setting the function

$$\psi_\alpha(t, x) = \frac{\sqrt{2\alpha}}{\cosh(x\sqrt{\alpha})} e^{i\alpha t}, \quad \alpha \in \mathbb{R}$$

solves (NLS) with initial value $u_0(x) = \psi_\alpha(0, x)$.

Hint: $\cosh(x) = (e^x + e^{-x})/2, \quad \frac{d}{dx}\cosh(x) = \sinh(x), \quad \cosh(x)^2 = 1 + \sinh(x)^2$.

For practical implementation issues we introduce periodic boundary conditions for the solution of (NLS), i.e. we restrict ourselves to the torus $\mathbb{T}_L := [-L\pi, L\pi]$ for some $L > 0$. More precisely, we replace $\mathbb{K} = \mathbb{T}_L$ in (NLS).

Now we choose L large enough, such that the boundary conditions are neglectable in the numerical solution.

Exercise 9:

Set $T = 1$, $L = 2$, $\alpha = 8$ and choose a time step size $\tau = 2^{-11}$. Furthermore set $u_0(x) = \psi_\alpha(0, x)$. Choose $N = 32$ grid points for spatial discretization.

a) In MATLAB implement the Strang splitting method applied to (NLS) for the space discretization with **spectral methods** and for the space discretization with **finite differences**.

b) Run both methods with $N_1 = 32$, $N_2 = 64$, $N_3 = 128$ and $N_4 = 256$ grid points and measure the elapsed time of each method for $N = N_\ell$, $\ell = 1, 2, 3, 4$, using the MATLAB `tic`,`toc` commands. What can you observe? Which method is "faster" and why?

c) Create an **order plot** for the spatial order of both methods using the values of N from above by comparing the numerical solution with the exact solution $\psi_\alpha(t, x)$ in the approximate L^2-norm, i.e.

$$err_{N_\ell} = \sqrt{h_\ell} \max_{t \in [0,T]} \left\| u^{num,N_\ell}(t) - \psi_\alpha^{N_\ell}(t) \right\|_{l^2}, \quad \ell = 1, 2, 3, 4,$$

where $u^{num,N_\ell}(t)$ is an array with approximation to $u(t, \cdot)$ in the grid points corresponding to N_ℓ, and where $\psi_\alpha^{N_\ell}(t)$ is an array with the values of the exact solution $\psi_\alpha(t, \cdot)$ in the grid points corresponding to N_ℓ, i.e.

$$u^{num,N_\ell}(t) = \left(u^{num,N_\ell}(t, x_j) \right)_{j=1}^{N_\ell} \quad \text{and} \quad \psi_\alpha^{N_\ell}(t) = \left(\psi_\alpha^{N_\ell}(t, x_j) \right)_{j=1}^{N_\ell}.$$

How does your plot change if you use a time step size $\tau = 2^{-12}$, $\tau = 2^{-13}$ or $\tau = 2^{-14}$ instead? Can you give an explanation?

d) Now reset $\tau = 2^{-11}$.

Create an **efficiency plot** for both methods, i.e. create a `loglog` plot showing the time which is needed to obtain a specific error.

Therefore on the X-axis insert the error, which each method produces at the number of grid points N_ℓ, $\ell = 1, 2, 3, 4$ and on the Y-axis put the corresponding elapsed time.

Which of the schemes shows a better efficiency?

Discussion in the problem class thursday 3:45 pm, in room 3.061 in the Kollegiengebäude Mathematik 20.30.