Karlsruher Institut für Technologie

Fakultät für Mathematik
Institut für Angewandte und
Numerische Mathematik

JProf. Dr. Katharina Schratz
Abdullah Demirel
SM · WS 13/14
12.11.2013

Splitting Methods, Blatt 2

Exercise 1:

Let Ω, $H \in \mathbb{R}^{n\times n}$ and $A, B \in C^1(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$. Show that

$a)$ the derivative of Ω^k with respect to H is given by

$$\partial_H \Omega^k = H\Omega^{k-1} + \Omega H \Omega^{k-2} + \ldots + \Omega^{k-2} H \Omega + \Omega^{k-1} H.$$

$b)$ a kind of „product rule"

$$\partial_H A(\Omega)B(\Omega) = \big(\partial_H A(\Omega)\big)B(\Omega) + A(\Omega)\big(\partial_H B(\Omega)\big)$$

can be obtained.

Reminder: The derivative of $A(\Omega)$ with respect to H is defined as

$$\partial_H A(\Omega) = \lim_{h\to 0} \frac{A(\Omega + hH) - A(\Omega)}{h}.$$

Exercise 2:

$a)$ Prove the following property of the adjoint operator

$$\Omega \operatorname{ad}_\Omega^k(H) = \operatorname{ad}_\Omega^k(H)\Omega + \operatorname{ad}_\Omega^{k+1}(H).$$

$b)$ Prove the following identity

$$\partial_H \Omega^k = \sum_{j=0}^{k-1} \binom{k}{j+1} \operatorname{ad}_\Omega^j(H)\Omega^{k-j-1}.$$

Reminder: Let Ω, $H \in \mathbb{R}^{n\times n}$, then the adjoint operator $\operatorname{ad}_\Omega(H)$ is defined as

$$\begin{aligned} &\operatorname{ad}_\Omega(H) := [\Omega, H] = \Omega H - H\Omega \\ \text{with}\quad &\operatorname{ad}_\Omega^0 = H, \quad \text{and} \quad \operatorname{ad}_\Omega^k(H) = \operatorname{ad}_\Omega^{k-1}([\Omega, H]). \end{aligned} \qquad (1)$$

Exercise 3:

Consider the following model problem

$$y'(t) = Ly(t) = (A+B)y(t), \quad y(0) = y_0 \in \mathbb{R}^n, \quad t \in [0, T],$$

where matrices $L, A, B \in \mathbb{R}^{n\times n}$.

Formulate and prove a theorem about the order of convergence of the Strang splitting method, if it is applied to this kind of model problem.

Exercise 4:

Let Ω, $H \in \mathbb{R}^{n\times n}$. Prove that

$a)$ the derivative of the exponential mapping is given by

$$\partial_H \exp(\Omega) = \operatorname{dexp}_\Omega(H)\exp(\Omega) \quad \text{with} \quad \operatorname{dexp}_\Omega(H) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \operatorname{ad}_\Omega^k(H).$$

$b)$ if the eigenvalues of ad_Ω are not equal to $2\pi i k$ with $k \in \mathbb{Z}_{\neq 0}$, then $d\exp_\Omega$ is invertible. For $\|\Omega\| \le \pi$ the inverse operator is given by

$$\operatorname{dexp}_\Omega^{-1}(H) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \operatorname{ad}_\Omega^k(H),$$

where B_k are the Bernoulli numbers defined by

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}.$$

Will be discussed in the exercise class on: 19.11.2013.