On this exercise sheet our aim is to construct a third order splitting method $\Psi_h^{(m)}$ for the model problem
\[ y = f^{(1)}(y) + f^{(2)}(y), \quad y(0) = y_0 \] (1)
with flows $\varphi_1^{[1]}$ and $\varphi_1^{[2]}$ corresponding to $f^{[1]}$ and $f^{[2]}$ respectively. We define
\[ \Psi_h^{(j)} = \varphi_{b,h}^{[2]} \circ \varphi_{a,h}^{[1]} \circ \Psi_h^{(j-1)}, \quad \Psi_h^{(0)} = Id. \]

In the lecture we proved the theorem:

The method $\Psi_h^{(m)}$ is consistent of order $p$ if $c_{1,m}^1 = c_{2,m}^1 = 1$ and $c_{\ell,m}^k = 0$ for $k = 2, \ldots, p$ and all $\ell$.

We found the following recurrences, $c_{\ell,0}^k = 0$, $\forall \ell, k$ and for $j \geq 1$:  
\begin{align*}
    c_{1,j}^1 &= c_{1,j-1}^1 + a_j \\
    c_{2,j}^1 &= c_{2,j-1}^1 + b_j \\
    c_{1,j}^2 &= c_{1,j-1}^2 + a_j b_j + c_{1,j-1}^1 b_j - c_{2,j-1}^1 a_j \\
    c_{1,j}^3 &= c_{1,j-1}^3 + a_j b_j + 2 c_{1,j-1}^1 a_j b_j - 3 c_{2,j-1}^2 a_j + (c_{1,j-1}^1)^2 b_j - c_{1,j-1}^1 c_{2,j-1}^1 a_j + c_{2,j-1}^2 a_j \\
    c_{2,j}^3 &= c_{2,j-1}^3 + a_j b_j - 4 c_{2,j-1}^2 a_j b_j + 3 c_{1,j-1}^1 b_j + (c_{2,j-1}^2)^2 a_j - c_{1,j-1}^1 c_{2,j-1}^1 b_j + c_{1,j-1}^1 b_j.
\end{align*}

Now we are looking for a method
\[ \Psi_h^{(3)}(y) = \varphi_{b,h}^{[2]} \circ \varphi_{a,h}^{[1]} \circ \varphi_{b,h}^{[1]} \circ \varphi_{a,h}^{[2]} \circ \varphi_{b,h}^{[1]} \circ \varphi_{a,h}^{[1]}(y), \]
where we choose the coefficients $a_j, b_j$, $j = 1, 2, 3$ such that $\Psi_h^{(3)}$ is of order $p = 3$, i.e. by the theorem above $a_j, b_j$, $j = 1, 2, 3$ have to satisfy
\[ c_{1,3}^1 = 1, \quad c_{2,3}^1 = 1, \quad c_{1,3}^2 = 0, \quad c_{2,3}^2 = 0. \] (2)

**Exercise 15:** (Third Order Splitting Scheme)

a) $\clubsuit$ Determine the expression for the coefficients $c_{\ell,3}^k$ from equation (2) in terms of $a_j, b_j$, $j = 1, 2, 3$, e.g. $c_{1,3}^1 = a_1 + a_2 + a_3$.

b) Determine the coefficients $a_j, b_j$, $j = 1, 2, 3$ such that (2) is satisfied.

**Hint:** For the coefficients we obtain a nonlinear system of equations. Since we have 6 unknowns but only 5 equations there will be many solutions. We can reduce the degrees of freedom by fixing one of the coefficients to a particular value. For example setting $a_1 = 1$, we find a simple solution.

c) MATLAB: Implement and test the method $\Psi_h^{(3)}$ (with the coefficients of part b)) applied to problem (1), where we assume $f^{[1]}$ and $f^{[2]}$ to be linear, i.e. $f^{[1]}(y) = Ay$, $f^{[2]}(y) = By$ for random complex matrices $A, B \in \mathbb{C}^{N \times N}$, $N = 20$. We furthermore choose random initial data $y_0 \in \mathbb{C}^N$. Create an order plot for $\Psi_h^{(3)}$ by comparing the numerical with the exact solution for various step sizes $h$.

**Exercise 16:** (Strang Splitting)

Verify that the coefficients of the Strang splitting method satisfy the order conditions from above up to $p = 2$.

Discussion in the problem class Thursday 15:45, in room 2.067 in the Kollegiengebäude Mathematik (20.30).

$\clubsuit$: Please try to do exercises marked with $\clubsuit$ at home.