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Wavelets – Theory and Applications (Winter 2008/09)

Lemma 3.30 and its proof

Lemma 3.30: *If* $\Gamma(\omega) = \sum_{k=N_1}^{N_2} \gamma_k \mathrm{e}^{-\imath k\omega}$ *with* $N_1 < N_2$ *and* $\sum_{k=N_1}^{N_2} \gamma_k = 1$ *then* $p(\omega) = \prod_{j=1}^{\infty} \Gamma(2^{-j}\omega)$ *is an entire function of exponential type.*

Especially, its inverse Fourier transform is a distribution compactly supported in $[N_1, N_2]$.

Proof: We rely on the Payley-Wiener theorem (see, e.g., W. Rudin: *Functional Analysis*. Tata McGraw-Hill, 1979):

> *If* u *is an entire function which is at most exponentially increasing, i.e., there are constants* $r,\ c,\ M \in \mathbb{R}$ *with*
>
> $$|u(\omega)| \leq c(1+|\omega|)^M \exp(r|\mathrm{Im}\omega|),$$
>
> *then* u *is the Fourier transform of a distribution with compact support in* $[-r, r]$.

We will prove an estimation of $|p(\omega)|$ for the case of symmetrically distributed coefficients $\{\gamma_k\}$, that is,

$$\Gamma(\omega) = \sum_{k=-N}^{N} \gamma_k \mathrm{e}^{-\imath k\omega}.$$

Define $\Gamma_1(\omega) := \mathrm{e}^{-\imath N\omega}\Gamma(\omega) = \sum_{k=0}^{2N} \gamma_{k-N}\mathrm{e}^{-\imath k\omega}$. Then,

$$p(\omega) = \prod_{j=1}^{\infty} \Gamma(2^{-j}\omega) = \prod_{j=1}^{\infty} \left(\mathrm{e}^{\imath N 2^{-j}\omega}\Gamma_1(2^{-j}\omega)\right) = \prod_{j=1}^{\infty} \mathrm{e}^{\imath N 2^{-j}\omega} \underbrace{\prod_{j=1}^{\infty} \Gamma_1(2^{-j}\omega)}_{=:\, p_1(\omega)}$$

$$= \mathrm{e}^{\imath N \sum_{j=1}^{\infty} 2^{-j}\omega} p_1(\omega) = \mathrm{e}^{\imath N\omega} p_1(\omega) = \mathrm{e}^{\imath N \mathrm{Re}\omega}\mathrm{e}^{-N\mathrm{Im}\omega} p_1(\omega)$$

and

$$|p(\omega)| \leq \mathrm{e}^{-N\mathrm{Im}\omega}|p_1(\omega)| \leq \mathrm{e}^{N|\mathrm{Im}\omega|}|p_1(\omega)|.$$

To apply the Payley-Wiener theorem we accordingly need to find only a polynomial bound for $|p_1(\omega)|$.

We have

$$|\Gamma_1(\omega) - 1| = |\Gamma_1(\omega) - \Gamma_1(0)| \leq \max_{0\leq\omega\leq 2\pi} |\Gamma_1'(\omega)|\,|\omega| = C_1\,|\omega|.$$

as well as

$$|\Gamma_1(\omega) - 1| \leq |\Gamma_1(\omega)| + 1 \leq 1 + \max_{0\leq\omega\leq 2\pi} |\Gamma_1(\omega)| = 1 + C_2.$$

Thus, $|\Gamma_1(\omega) - 1| \le C_\Gamma \min\{1, |\omega|\}$ and

$$|\Gamma_1(\omega)| \le 1 + C_\Gamma \min\{1, |\omega|\}.$$

If $|\omega| \le 1$ then

$$\begin{aligned} |p_1(\omega)| &\le \prod_{j=1}^{\infty}(1 + C_\Gamma 2^{-j}|\omega|) \le \prod_{j=1}^{\infty}(1 + C_\Gamma 2^{-j}) \\ &\le \prod_{j=1}^{\infty}\exp(C_\Gamma 2^{-j}) = \exp\left(C_\Gamma \sum_{j=1}^{\infty} 2^{-j}\right) = \exp(C_\Gamma). \end{aligned}$$

If $|\omega| > 1$ then there exists $j_0 = j_0(\omega) \ge 1$ such that $2^{j_0-1} < |\omega| \le 2^{j_0}$ and we have

$$\begin{aligned} |p_1(\omega)| &= \prod_{j=1}^{j_0-1}\Gamma_1(\underbrace{2^{-j}\omega}_{|\cdot|>1}) \prod_{j=j_0}^{\infty}\Gamma_1(2^{-j}\omega) \le (1 + C_\Gamma)^{j_0-1}\prod_{j=0}^{\infty}\Gamma_1(2^{-j}\underbrace{2^{-j_0}\omega}_{|\cdot|\le 1}) \\ &\le (1 + C_\Gamma)^{j_0-1}\exp(C_\Gamma) \le (1 + C_\Gamma)^{\log_2|\omega|}\exp(C_\Gamma) = |\omega|^{\log_2(1+C_\Gamma)}\exp(C_\Gamma). \end{aligned}$$

Combining both estimates for $|p_1|$ yields the polynomial bound

$$|p_1(\omega)| \le \exp(C_\Gamma)\big(1 + |\omega|\big)^{\lfloor \log_2(1+C_\Gamma)\rfloor + 1}.$$

In the general situation $\{\gamma_k\}_{N_1 \le k \le N_2}$, $N_1 \ne -N_2$, consider $\exp(\imath \frac{N_1+N_2}{2}\omega)p_1(\omega)$. The inverse Fourier transform then effects a shift of $(N_1 + N_2)/2$. ✓