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Wavelets – Theory and Applications (Winter 2008/09)

3.3 Wavelet-Bases (Summary)

Definition 3.11: *A function $\psi \in L^2(\mathbb{R})$ is called a (dyadic)* ***basis-wavelet*** *if*

a) *any $f \in L^2(\mathbb{R})$ can be written in a wavelet series*

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k} \tag{3.8}$$

with coefficients $\{d_{j,k}\}_{j,k\in\mathbb{Z}}$ where $\psi_{j,k}(\cdot) = 2^{j/2}\psi(2^j \cdot -k)$, and

b) *there exist positive constants A and B such that*

$$A \sum_{j,k\in\mathbb{Z}} |d_{j,k}|^2 \leq \Big\| \sum_{j,k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k} \Big\|^2_{L^2(\mathbb{R})} \leq B \sum_{j,k\in\mathbb{Z}} |d_{j,k}|^2. \tag{3.9}$$

Put differently, the function $\psi \in L^2(\mathbb{R})$ is a basis-wavelet if the family $\{\psi_{j,k} \colon j,k \in \mathbb{Z}\}$ constitutes a Riesz-basis for $L^2(\mathbb{R})$.

Lemma 3.12: *Let ψ be a basis-wavelet. Then, the series representation* (3.8) *is unique:*

$$\sum_{j,k\in\mathbb{Z}} d_{j,k}\psi_{j,k} = \sum_{j,k\in\mathbb{Z}} \widetilde{d}_{j,k}\psi_{j,k} \quad \Longleftrightarrow \quad d_{j,k} = \widetilde{d}_{j,k} \ \forall\, j,k \in \mathbb{Z}.$$

The **discrete wavelet transform** (DWT) with respect to the basis-wavelet ψ maps $f \in L^2(\mathbb{R})$ (or sampled values of f) to its expansion coefficients $\{d_{j,k}(f)\}_{j,k\in\mathbb{Z}}$ from (3.8).

Lemma 3.13: *Let $\{\varphi_k\}_{k\in\Gamma}$ (Γ countable set) be a Riesz-basis of the Hilbert space X with bounds A and B. Then, $\{\varphi_k\}_{k\in\Gamma}$ is also a frame in X with the same bounds.*

The admissibility condition (2.1) is not required in the definition of a basis-wavelet (compare definition of a wavelet frame). However, if ψ is a basis-wavelet then $(\psi, 2, 1)$ generates a frame according to the above lemma. Thus, in view of Lemma 3.8, ψ satisfies (2.1) and the notations basis-wavelet and wavelet-series are justified subsequently.

Define operators $P_l \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $Q_l \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$P_l f := \sum_{j=-\infty}^{l-1} \sum_{k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k} \quad \text{and} \quad Q_j f := \sum_{k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k},$$

respectively. Then,

$$P_l = \sum_{j=-\infty}^{l-1} Q_j \quad \text{and} \quad \mathrm{Id} = P_l + \sum_{j=l}^{\infty} Q_j.$$

Lemma 3.14: *Let ψ be a basis-wavelet. Then, the operators $P_l\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $Q_l\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are linear and continuous projection operators: $P_l = P_l^2$ and $Q_l = Q_l^2$. Moreover,*

i) $P_{l+1} = P_l + Q_l$ *and* $\lim_{l\to\infty} P_l f = f$ *for all* $f \in L^2(\mathbb{R})$,

ii) $P_{l+1}P_l = P_l$ *and* $P_{l+1}Q_l = Q_l$.

Lemma 3.15: *Let ψ be a basis-wavelet and define subspaces of $L^2(\mathbb{R})$ by*

$$V_l = P_l L^2(\mathbb{R}) \quad \text{and} \quad W_l = Q_l L^2(\mathbb{R}).$$

Then,

i) $V_l \subset V_{l+1}$, $V_l \oplus W_l = V_{l+1}$ *(direct sum: $V_l \cap W_l = \{0\}$)*,

ii) $\mathrm{closure}_{L^2}\big(\bigcup_{l\in\mathbb{Z}} V_l\big) = L^2(\mathbb{R})$, $\bigcap_{l\in\mathbb{Z}} V_l = \{0\}$.

Theorem 3.16: *Let ψ be a basis-wavelet. Further, assume that there exists a function $\varphi \in L^2(\mathbb{R})$ such that P_l, $l \in \mathbb{Z}$, can be written as*

$$P_l f = \sum_{k\in\mathbb{Z}} c_{l,k}(f)\varphi_{l,k}$$

with uniquely determined coefficients $\{c_{l,k}\}_{k\in\mathbb{Z}}$ where $\varphi_{l,k}(\cdot) = 2^{l/2}\varphi(2^l \cdot - k)$.

Then, there are uniquely determined sequences $h = \{h_k\}_{k\in\mathbb{Z}}$, $g = \{g_k\}_{k\in\mathbb{Z}}$, $\widetilde{h} = \{\widetilde{h}_k\}_{k\in\mathbb{Z}}$, and $\widetilde{g} = \{\widetilde{g}_k\}_{k\in\mathbb{Z}}$ such that

$$\varphi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\varphi(2x-k) = \sum_{k\in\mathbb{Z}} h_k\varphi_{1,k}(x) \qquad \text{(scaling equation)},$$

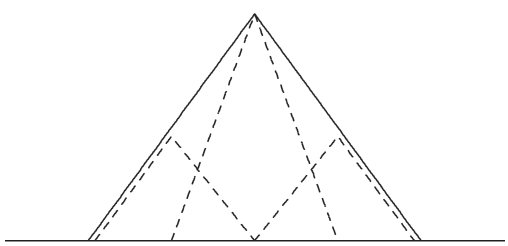
$$\psi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} g_k\varphi(2x-k) = \sum_{k\in\mathbb{Z}} g_k\varphi_{1,k}(x) \qquad \text{(wavelet equation)},$$

$$\sqrt{2}\varphi(2x-n) = \sum_{k\in\mathbb{Z}} \big(\widetilde{h}_{n-2k}\varphi(x-k) + \widetilde{g}_{n-2k}\psi(x-k)\big) \qquad \text{(decomposition equation)}.$$

Moreover,

$$\sum_{k\in\mathbb{Z}} \big(\widetilde{h}_{n-2k}h_{r-2k} + \widetilde{g}_{n-2k}g_{r-2k}\big) = \begin{cases} 1 & : \ n = r, \\ 0 & : \ \textit{otherwise}. \end{cases}$$

As φ satisfies a scaling or refinement equation it is called a *scaling function*. A scaling function can be written as a superposition of scaled and translated versions of itself:



The wavelet equation expresses ψ as sum of scaling functions. Starting out from a scaling function we expect good chances to find a corresponding basis-wavelet. In a first step to construct basis-wavelets we therefore study scaling equations.