Karlsruhe Institute of Technology

Prof. Dr. Andreas Rieder
Dipl.-Math. techn. Tim Kreutzmann
Institute for Applied and Numerical Analysis
Department of Mathematics

Wavelets
Winter Semester 2013/2014

Problem Sheet 1 of October 28, 2013

Exercise 1:
Let $F\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the Fourier transform. Verify that:

(a) $FT^b = E^b F$ for $b \in \mathbb{R}^d$, where T^b and E^b are defined by

$$T^b f(x) = f(x-b) \quad \text{and} \quad E^b f(x) = \mathrm{e}^{-ib\cdot x} f(x)\,.$$

(b) $FD^a = D^{1/a}F$ for $a \in \mathbb{R} \setminus \{0\}$, where D^c is the dilation operator

$$D^c f(x) = |c|^{-d/2} f(c^{-1}x)\,.$$

Exercise 2:
Show that $C_0^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$.

Hint: For $f \in L^1(\mathbb{R}^d)$ with compact support and $\varepsilon > 0$ consider the function

$$f_\varepsilon = \int_{\mathbb{R}^d} g_\varepsilon(\cdot - y) f(y)\,\mathrm{d}y\,,$$

where $g_\varepsilon = \varepsilon^{-d} g(\cdot/\varepsilon)$ with $g \in C_0^\infty(\mathbb{R}^d)$ such that

$$\operatorname{supp} g \subset B_1(0)\,, \quad g \geq 0\,, \quad \int_{\mathbb{R}^d} g\,\mathrm{d}x = 1\,.$$

Moreover, use the following statement of the Kolmogorov compactness criterion: If $M \subset L^1(\mathbb{R}^d)$ is relatively compact, then

$$\lim_{h\to 0} \int_{\mathbb{R}^d} \big|\phi(x+h) - \phi(x)\big|\,\mathrm{d}x = 0$$

uniformly for $\phi \in M$.

Exercise 3:
Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\sin(nx)\,\mathrm{d}x = 0\,.$$

Exercise 4:
Let $a > 0$ and $b \in \mathbb{R}$. Compute the Fourier transform of the function

$$f\colon \mathbb{R} \to \mathbb{C}\,, \quad f(t) = \exp\big(-(a-ib)t^2\big)\,.$$

Hint: Derive in a first step that

$$\hat{f}'(\omega) + \frac{\omega}{2(a-ib)}\hat{f}(\omega) = 0\,.$$

These excersises are discussed in the problem class on **Thursday, October 31, 2013.**