Karlsruhe Institute of Technology

Prof. Dr. Andreas Rieder
Dipl.-Math. techn. Tim Kreutzmann
Institute for Applied and Numerical Analysis
Department of Mathematics

Wavelets
Winter Semester 2013/2014

Problem Sheet 10 of January 13, 2014

Exercise 29:
Let the signal $s \in L^2(\mathbb{R})$ be Hölder continuous of order $\alpha \in]0,1]$, i.e., there exists a $C_H > 0$ with

$$|s(x) - s(y)| \le C_H |x-y|^\alpha \quad \text{for all } x, y \in \mathbb{R}\,.$$

Let $B\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support that satisfies

$$\sum_{k\in\mathbb{Z}} B(\,\cdot\, - k) = 1\,.$$

We define the function f by

$$f(\,\cdot\,) = \sum_{k\in\mathbb{Z}} s(hk) B(\,\cdot\, - k)$$

with $h > 0$. Show that

$$\|f(\,\cdot\,) - s(h\,\cdot\,)\|_{L^\infty} \le C h^\alpha\,,$$

where C is a positive constant.

Exercise 30:
Let $v \in \ell^1(\mathbb{Z})$ be real-valued. Moreover, let $h, \widetilde{h}, g$ and $\widetilde{g}$ be perfect reconstruction filters. Verify the following representation of v with $j \in \mathbb{N}_0$:

$$v = (S^\uparrow)^{j+1} S^\downarrow \widetilde{h} * R (S^\uparrow)^j h * v + (S^\uparrow)^{j+1} S^\downarrow \widetilde{g} * R (S^\uparrow)^j g * v\,.$$

Herein, the operators $R, S^\downarrow$ and $S^\uparrow$ are given as in Exercise 28.

Exercise 31:
Let ψ be a real-valued wavelet such that there exist positive constants A and B with

$$\frac{A}{2\pi} \le \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\omega)|^2 \le \frac{B}{2\pi} \quad \text{for all } \omega \in \mathbb{R}\,. \tag{1}$$

(a) Verify that

$$A\|f\|^2_{L^2(\mathbb{R})} \le \sum_{j\in\mathbb{Z}} 2^{-j} \|\sqrt{c_\psi} W_\psi f(2^j,\cdot)\|^2_{L^2(\mathbb{R})} \le B \|f\|^2_{L^2(\mathbb{R})}\,.$$

(b) Show: If $\varphi \in L^2(\mathbb{R})$ satisfies

$$\sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}}(2^j\omega)\, \widehat{\varphi}(2^j\omega) = \frac{1}{2\pi} \quad \text{for all } \omega \in \mathbb{R}\setminus\{0\}\,, \tag{2}$$

then (formally)

$$f = \sqrt{c_\psi} \sum_{j\in\mathbb{Z}} 2^{-j} W_\psi f(2^j,\cdot) * D^{2^j} \varphi\,.$$

(c) Starting from (1), construct a $\varphi \in L^2(\mathbb{R})$ satisfying (2).

These excersises are discussed in the problem class on **Thursday, January 16, 2014**.