Karlsruhe Institute of Technology

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Wavelets
Winter Semester 2013/2014

Problem Sheet 12 of January 27, 2014

Exercise 35:
Show that the Gaußian $\varphi(t) = \mathrm{e}^{-t^2/2}$ is not a scaling function.

Exercise 36:
A scaling function φ is called *interpolating scaling function* if $\varphi(0) = 1$ and $\varphi(m) = 0$ for $m \in \mathbb{Z}\backslash\{0\}$.
Show that the symbol H of an interpolating scaling function satisfies

$$H(\omega) + H(\omega + \pi) = 1\,.$$

Exercise 37:
Prove the following properties of cardinal B-splines $B_N = B_{N-1} * \chi_{[0,1]}$, $N \geq 2$, with $B_1 = \chi_{[0,1]}$:

(a) $B_N(t) = \displaystyle\int_{t-1}^{t} B_{N-1}(x)\,\mathrm{d}x\,, \quad N \geq 2\,.$

(b) $B_N'(t) = B_{N-1}(t) - B_{N-1}(t-1)\,, \quad N \geq 2\,.$

(c) $B_N(t) = \dfrac{t}{N-1} B_{N-1}(t) + \dfrac{N-t}{N-1} B_{N-1}(t-1)\,, \quad N \geq 2\,.$

(d) For $N \in \mathbb{N}$ and for any $k \in \mathbb{Z}$, there exists a $p_k \in \Pi_{N-1}$ such that $B_N|_{[k,k+1]} = p_k|_{[k,k+1]}\,.$

(e) $B_N(\,\cdot\,) = 2^{1-N} \displaystyle\sum_{k=0}^{N} \binom{N}{k} B_N(2\,\cdot\, - k)\,, \quad N \in \mathbb{N}\,.$

(f) $\displaystyle\sum_{k\in\mathbb{Z}} B_N(\,\cdot\, - k) = 1\,, \quad N \in \mathbb{N}\,.$

(g) $\displaystyle\int_{\mathbb{R}} B_N(t)\mathrm{d}t = 1\,, \quad N \in \mathbb{N}\,.$

(h) $\displaystyle\int_{\mathbb{R}} B_N(t) B_N(t-x)\mathrm{d}t = B_{2N}(N+x)\,, \quad N \in \mathbb{N}\,.$
Hint: Use that $B_N * B_m = B_{N+m}$ and $B_N(x) = B_N(N-x)\,.$

(i) $\displaystyle\sum_{k\in\mathbb{Z}} |\widehat{B}_N(\omega + 2\pi k)|^2 = \frac{1}{2\pi} \sum_{n=-N+1}^{N-1} B_{2N}(N+n)\mathrm{e}^{-in\omega}\,, \quad N \in \mathbb{N}\,.$
Hint: Use part (h), Parseval's identity and $\operatorname{supp} B_{2N} = [0, 2N]\,.$

(j) $\widehat{B}_N(\omega) = \dfrac{1}{\sqrt{2\pi}} \operatorname{sinc}^N\left(\dfrac{\omega}{2}\right) \mathrm{e}^{iN\omega/2}\,, \quad N \in \mathbb{N}\,.$

These excersises are discussed in the problem class on **Thursday, January 30, 2014**.