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Wavelets
Winter Semester 2013/2014

Problem Sheet 2 of November 04, 2013

Exercise 5:
Verify that the Hermite polynomials $H_n\colon \mathbb{R} \to \mathbb{R}$ defined by

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}, \quad n \in \mathbb{N}_0,$$

satisfy the recursion formula

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \in \mathbb{N}.$$

Exercise 6:
Let H_n be the Hermite polynomial of degree $n \in \mathbb{N}_0$. Prove that the function

$$h_n(x) = H_n(x)\mathrm{e}^{-x^2/2}$$

is an eigenfunction of the Fourier transform with eigenvalue $(-i)^n$, that is,

$$\widehat{h_n} = (-i)^n h_n.$$

Hint: Use the recursion formula and an inductive argument.

Exercise 7:
Show that for $d \in \mathbb{N}$ the identity

$$\int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/2} \,\mathrm{d}x = (2\pi)^{d/2}$$

holds.

Hint: Start calculating $\left(\int_{\mathbb{R}} \mathrm{e}^{-t^2/2} \,\mathrm{d}t\right)^2$.

Exercise 8:
Let $h \in L^2(\mathbb{R}^2)$ and $\varphi \in L^2(\mathbb{R}) \setminus \{0\}$ with $\|\varphi\|_{L^2} = 1$. Show the following statement:
There exists an $f \in L^2(\mathbb{R}^2)$ such that $h = F_\varphi f$ if and only if

$$h(\xi_0, b_0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} h(\xi, b) K(\xi_0, \xi, b_0, b) \,\mathrm{d}\xi \,\mathrm{d}b,$$

where

$$K(\xi_0, \xi, b_0, b) = \langle E^{-\xi} T^b \varphi, E^{-\xi_0} T^{b_0} \varphi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \varphi(t-b)\overline{\varphi}(t-b_0)\mathrm{e}^{-i(\xi_0-\xi)t} \,\mathrm{d}t.$$

These excersises are discussed in the problem class on **Thursday, November 07, 2013**.