Prof. Dr. Andreas Rieder
Dipl.-Math. techn. Tim Kreutzmann
Institute for Applied and Numerical Analysis
Department of Mathematics

Wavelets
Winter Semester 2013/2014

Problem Sheet 6 of December 02, 2013

Exercise 17:
Let X be a Banach space and Y be normed space. For the bounded linear operator $F\colon X \to Y$ let there exist a positive constant α such that

$$\|Fx\|_Y \geq \alpha \|x\|_X \quad \text{for all } x \in X\,.$$

Show that the range $\mathsf{R}(F)$ of F is closed and $F\colon X \to \mathsf{R}(F)$ is continuously invertible with

$$\|F^{-1}|_{\mathsf{R}(F)}\| \leq \alpha^{-1}\,.$$

Exercise 18:
Let $S\colon X \to X$ be a bounded linear operator on the Hilbert space X satisfying

$$\langle Sx, x\rangle_X \geq \alpha \|x\|_X^2 \quad \text{for all } x \in X\,,$$

where α is a positive constant. Show that S is continuously invertible with

$$\|S^{-1}\| \leq \alpha^{-1}\,.$$

Hint: To verify surjectivity of S, consider the mapping $\Psi\colon X \to X$, $\Psi(x) = x - \kappa(Sx - y)$ for $y \in X$ and $\kappa > 0$. Show that $\|\Psi(x) - \Psi(z)\|_X^2 \leq (1 - 2\kappa\alpha + \kappa^2\|S\|^2)\|x - z\|_X^2$. In particular, Ψ is a contraction if $\kappa \in]0, 2\alpha/\|S\|^2[$. Apply now Banach's fixed point theorem.

Exercise 19:
Let $\{\varphi_k\}_{k\in\Gamma}$ and $\{\widetilde{\varphi}_k\}_{k\in\Gamma}$ be two frames in H being dual to each other. Verify: If $f = \sum_{k\in\Gamma} c_k\varphi_k$ for $c = \{c_k\}_{k\in\Gamma} \in \ell^2(\Gamma)$ and if not all c_k equal $\langle f, \widetilde{\varphi}_k\rangle_H$, then

$$\sum_{k\in\Gamma} |c_k|^2 > \sum_{k\in\Gamma} |\langle f, \widetilde{\varphi}_k\rangle_H|^2\,.$$

Hint: Observe that $F^*c = f$ and split $c = a + b$ with $a \in \mathsf{R}(F) = \mathsf{R}(\widetilde{F})$ and $b \in \mathsf{R}(F)^\perp = \mathsf{N}(F^*)$, where F and $\widetilde{F}$ are the frame and dual frame operators, respectively.

These excersises are discussed in the problem class on **Thursday, December 05, 2013**.