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Wavelets
Winter Semester 2013/2014

Problem Sheet 8 of December 16, 2013

Exercise 23:
Let ψ be a basis wavelet. For $f \in L^2(\mathbb{R})$ we define

$$P_l f = \sum_{j=-\infty}^{l-1} \sum_{k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k} \quad \text{and} \quad Q_j f = \sum_{k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k}\,.$$

Show that P_l and Q_j are linear bounded projection operators in $L^2(\mathbb{R})$.

Exercise 24:
Let $\varphi = \chi_{[0,1]}$ and let ψ be the Haar wavelet. Determine the unique finite sequences $\{h_k\}_{k\in\mathbb{Z}}$, $\{g_k\}_{k\in\mathbb{Z}}$, $\{\widetilde{h}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{g}_k\}_{k\in\mathbb{Z}}$ such that for almost all $x \in \mathbb{R}$:

$$\begin{aligned}
\varphi(x) &= \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \varphi(2x-k)\,,\\
\psi(x) &= \sqrt{2}\sum_{k\in\mathbb{Z}} g_k \varphi(2x-k)\,,\\
\sqrt{2}\varphi(2x-n) &= \sum_{k\in\mathbb{Z}} \left(\widetilde{h}_{n-2k}\varphi(x-k) + \widetilde{g}_{n-2k}\psi(x-k)\right), \quad n\in\mathbb{Z}\,.
\end{aligned}$$

In addition, verify that the Haar wavelet is a basis wavelet. More precisely, show that $\mathcal{H} = \{\psi_{j,k} \colon j,\, k \in \mathbb{Z}\}$ constitutes an orthonormal basis in $L^2(\mathbb{R})$.
Hint: For the completeness of $\mathcal{H}$ show that the orthogonal complement of $\mathcal{H}$ is $\{0\}$. To this end, use that piecewise constant functions are dense in $L^2(\mathbb{R})$.

Exercise 25:
Let the signal $s \in L^2(\mathbb{R})$ be Hölder continuous of order $\alpha \in]0,1[$, i.e., there exists a $C_H > 0$ with

$$|s(x) - s(y)| \le C_H |x-y|^\alpha \quad \text{for all } x, y \in \mathbb{R}\,.$$

Let φ be a continuous function with compact support that satisfies

$$\sum_{k\in\mathbb{Z}} \varphi(\cdot - k) = 1\,.$$

We define the function f by

$$f(t) = \sum_{k\in\mathbb{Z}} s(hk)\varphi(h^{-1}t - k) \quad \text{for } t \in \mathbb{R}$$

with $h > 0$. Show that

$$|s(hl) - f(hl)| \le C_H \left(\sum_{k\in\mathbb{Z}} |k|^\alpha |\varphi(k)|\right) h^\alpha$$

for all $l \in \mathbb{Z}$.

These excersises are discussed in the problem class on **Thursday, December 19, 2013**.

We wish you a merry Christmas and
a happy New Year 2014 !