A Data Transformation Method in Electrical Impedance Tomography

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Introduction

To electrical impedance tomography an elliptic boundary value problem is fundamental. Since the numerical solution of the corresponding nonlinear inverse 3D problem is both time- and memory-consuming, one often switches to the reconstruction in cross sections in real applications and assume some symmetry for the explored material. This poster presents a novel approach, which solves a cylindrical 3D problem in a very cheap way by reduction to a 2D problem. The idea is based on a transformation of the measured data sets and the subsequent use of a 2D reconstruction algorithm. Although the transformation is not exact, the approach provides good results for a class of material property distributions even for complex valued data, comparable to common approaches.

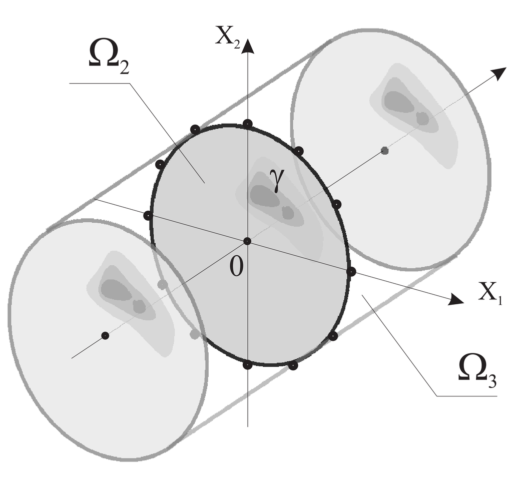

Figure 1: 3D and 2D model setting

$$\mathrm{div}(\gamma\nabla u) = 0 \quad \text{in } \Omega_2\times\mathbb{R} =: \Omega_3 \subset \mathbb{R}^3, \tag{1a}$$
$$\gamma(\nu\cdot\nabla u) = f(x_1,x_2)\delta(x_3) \quad \text{on } \partial\Omega_3 \tag{1b}$$
$$\int_{\partial\Omega_3} f(x_1,x_2)\delta(x_3)ds = 0, \tag{1c}$$

$$-\mathrm{div}(\gamma\nabla\tilde{u}) + \omega^2\gamma\tilde{u} = 0 \quad \text{in } \Omega_2, \quad \omega\in\mathbb{R} \tag{2a}$$
$$\gamma(\nu\cdot\nabla\tilde{u}) = f \quad \text{on } \partial\Omega_2, \tag{2b}$$
$$\int_{\partial\Omega_2} f ds = 0, \tag{2c}$$

Solving the forward problem is an important part of an iterative reconstruction algorithm. In figure 2 we give a comparison for this step dependent on the dimension of the problem and the conductivity distribution: (a) 3D forward problem, (b) 3D cylindrical forward problem, so-called 2.5D problem [2] and (c) 2D forward problem.

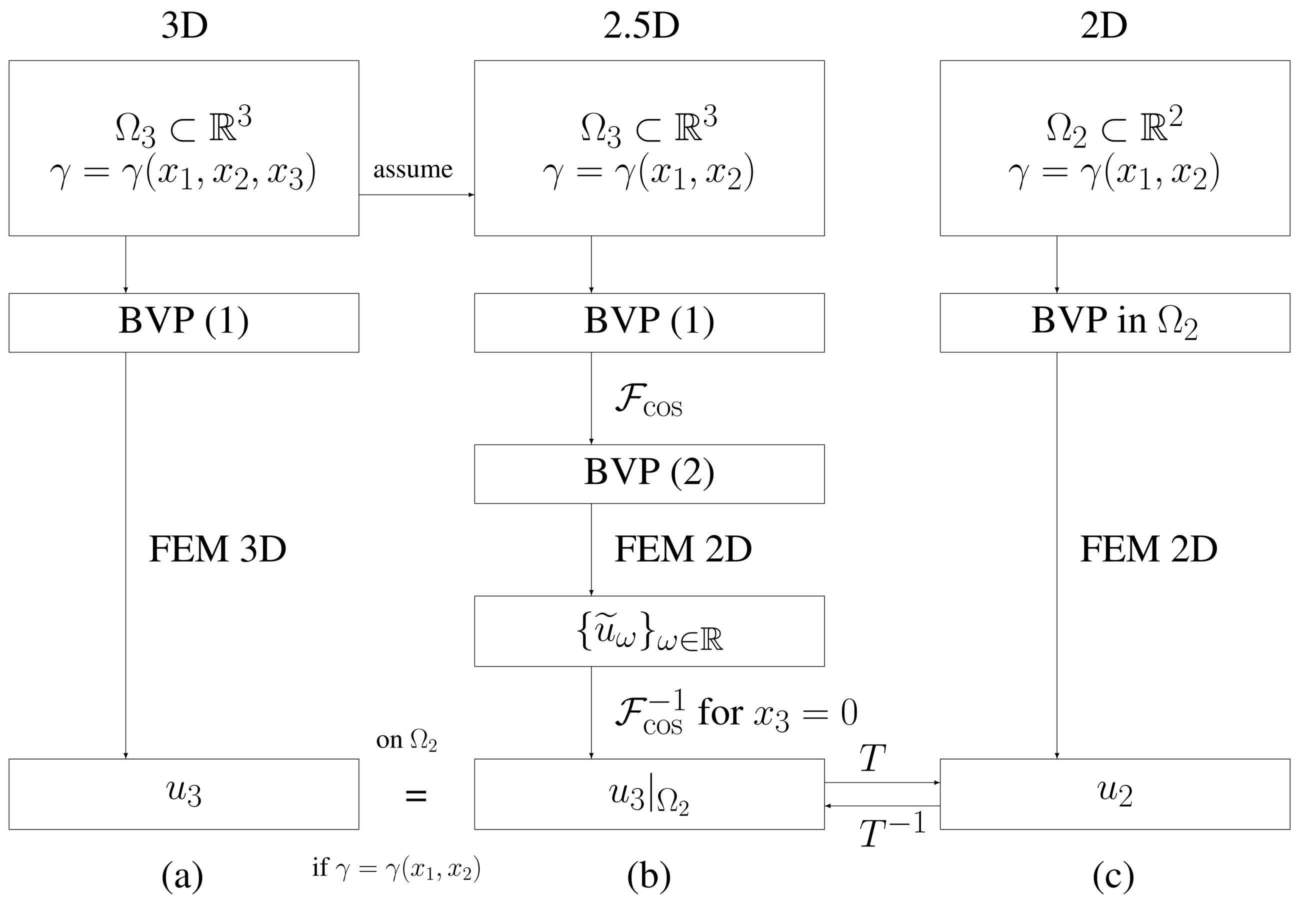

Figure 2: Comparison of 3D, 2.5D and 2D forward problem. The data transformation T represents a bridge between the 3D procedure in (b) and the 2D procedure in (c).

The Data Transformation Idea

The idea of the data transformation method consists of the adaptation of a real 3D measurement set to the forward solution of a 2D problem using a certain mapping (3). Afterwards we can perform a 2D reconstruction from measurement data set. The basis for this adaptation/transformation are the fundamental solutions of 2D and 3D Laplacian

$$\Phi_2(x,y) := -\frac{1}{2\pi}\ln|x-y|, \quad x,y\in\mathbb{R}^2, \quad \Phi_3(x,y) := \frac{1}{4\pi}\frac{1}{|x-y|}, \quad x,y\in\mathbb{R}^3, \quad x\neq y.$$

Because of the boundary conditions we have to operate with Green functions which are

$$\mathcal{G}_2(x,0) = \Phi_2(x,0) + w_2(x), \quad x\in\Omega_2\subset\mathbb{R}^2, \qquad \mathcal{G}_3(x,0) = \Phi_3(x,0) + w_3(x), \quad x\in\Omega_3\subset\mathbb{R}^3,$$

where w_2, w_3 are smooth functions arising from the Neumann boundary conditions. Thus, for the case of a homogeneous conductivity distribution $\gamma\equiv 1$ the potential distribution for $x\in\overline{\Omega}_n$, $n\in\{2,3\}$ can be determined as a *modified single layer potential* by

$$S_n f_n(x) := \begin{cases} H_\diamond^{-1/2}(\partial\Omega_n) \to H_\diamond^1(\Omega_n) \\ f_n(x) \mapsto u_n^0(x) := \int_{\partial\Omega_n} \mathcal{G}_n(x,y)f_n(y)\,ds_y \end{cases}$$

which we use for the following definition.

Definition 1. *The Data Transformation is given by*

$$T := \begin{cases} H_\diamond^{1/2}(\partial\Omega_2) \to H_\diamond^{1/2}(\partial\Omega_2) \\ u_3|_{\partial\Omega_2} \mapsto hu_3|_{\partial\Omega_2} \end{cases}, \quad \textit{where} \quad h(x) := \frac{S_2f_2(x)}{S_3f_3(x)}, \quad x\in\partial\Omega_2. \tag{3}$$

Data Transformation Method Procedure

1. Let a voltage measurement data set V_3 of the 3D forward problem (1) be given
2. Estimate the mean apparent conductivity distribution γ_0
3. Compute the Green's functions G_2 and G_3 for the given geometry and the homogeneous conductivity distribution γ_0
4. Transform the measurement data set V_3 according to definition (3), $V_2 := T(V_3)$
5. Use the transformed data set V_2 as input for the reconstruction algorithm for a 2D problem

The reconstruction accuracy can be improved by putting the steps 2-5 into a loop and by replacing the homogeneous conductivity by the interim solution γ_k.

Analytical Results

The introduced data transformation (3) is exact only for homogeneous conductivity distributions. Hence we are interested in an estimation for the error $u_2 - T(u_3)$. Consider a transmission EIT problem, i.e. assume thet there is an inclusion D_2 in a Lipschitz domain Ω_2 with $d = \mathrm{dist}(D_2,\partial\Omega)$ and $\gamma = 1 + b\chi_D$, $b\geq 0$. Then using the power gap $W - W_D$ with the integrals

$$W := \int_{\partial\Omega} fu\,dx \quad \text{and} \quad W_D := \int_{\partial\Omega} fu_D\,dx$$

we can give the lower bound for the size of an inclusion D_2

$$\frac{1}{c}\frac{W-W_D}{W} \leq |D_2|,$$

where c is a constant, compare [1]. For a class of symmetric domains Ω_2 and symmetric current patterns f this inequality is used to prove the statement

$$\|u_2 - T(u_3)\|_f \leq b(1+C)\int_{D_2}|\nabla u_2^0(x)|^2dx, \tag{4}$$

with $C = C(d,\gamma,\partial\Omega)$. Moreover, if Ω_2 is a disc, the modified single layer potentials can be written in analytical form, (see figure 3). Thus the data transformation as well. In this case the reconstruction algorithm is essentially faster.

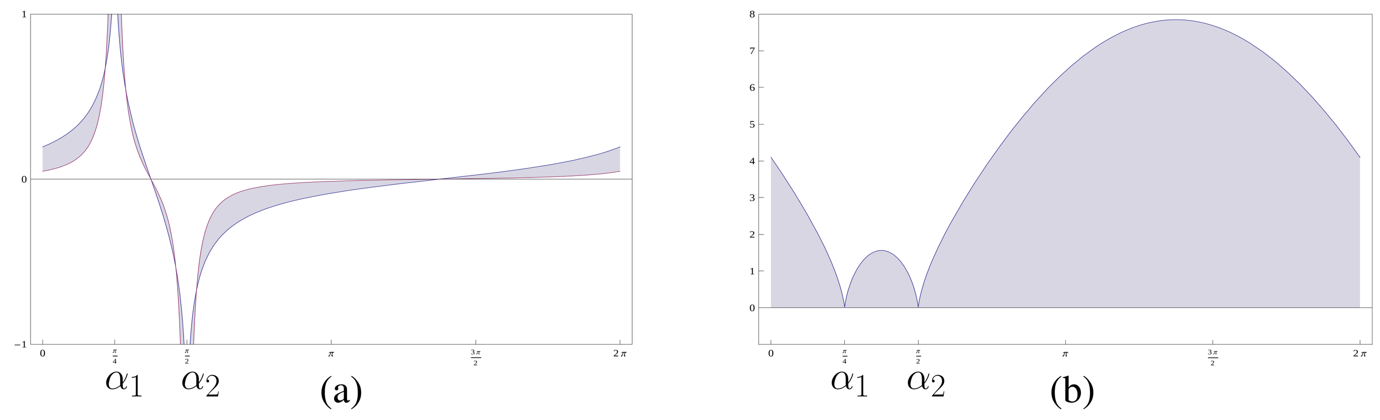

Figure 3: (a) Analytical trace $u_3|_{\partial\Omega_2}$ for a cylinder (red) and $u_2|_{\partial\Omega_2}$ for a disc (blue); (b) Corresponding multiplier function h defined in (3); Applied current pattern: source at $\alpha_1 = \pi/4$ and sink at $\alpha_2 = \pi/2$

Numerical Example

We show the performance of the transformation on one numerical example. In order to avoid a so-called *inverse crime*, we used different finite element meshes for the computation of the synthetic data of the forward problem, and for the reconstruction algorithm. We investigate a thorax model including heart, lung and bone. The corresponding mean conductivities of the tissues are taken out from medical literature. Here we provide the heart tissue conductivity with an imaginary part to show that the presented method can handle complex-valued conductivity as well.

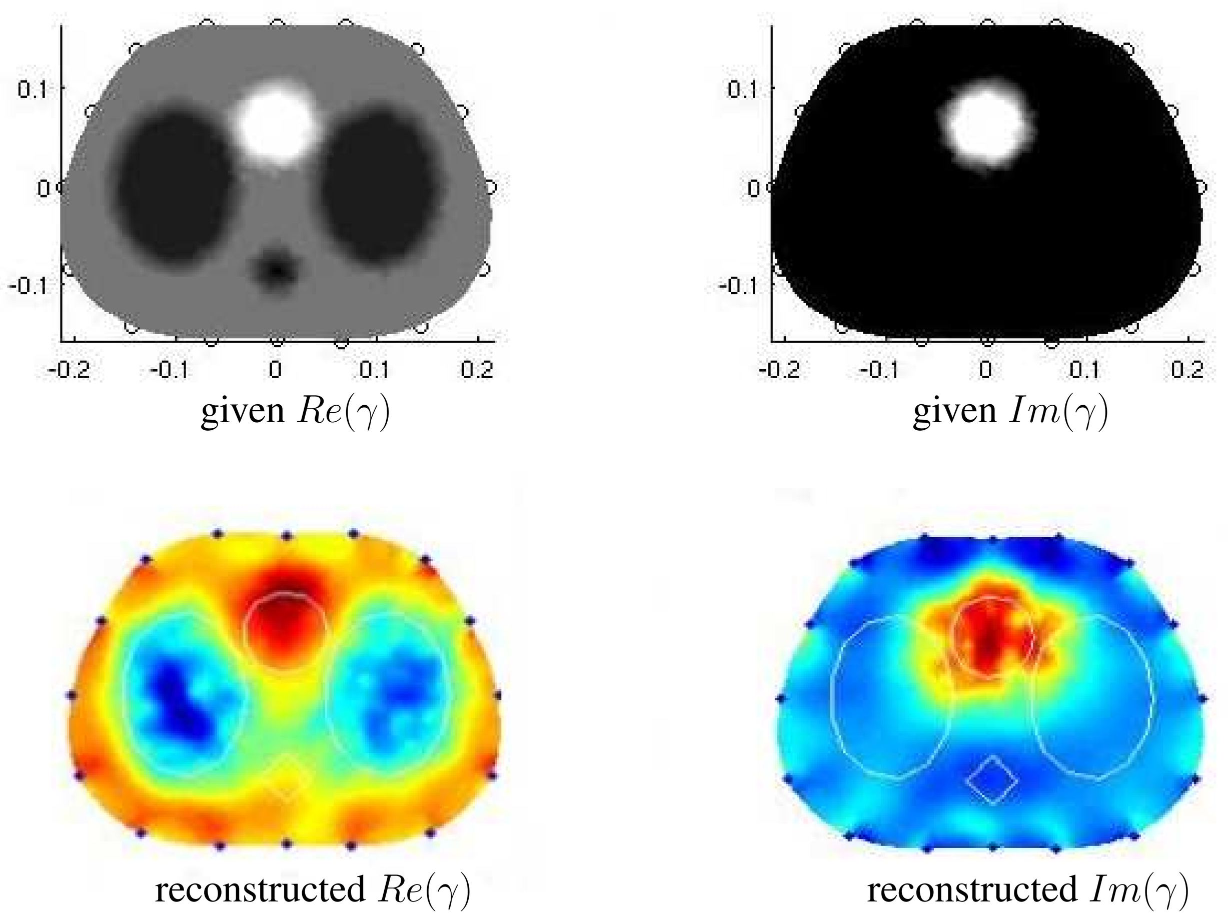

Figure 4: Reconstruction result for thorax model including heart, lungs and backbone. 16 electrodes and noise free measuremen data as dipole-dipole configuration are applied. Reconstruction algorithms: Tikhonov-regularised Gauss-Newton method

The results achieved are qualitatively comparable with common reconstructions. In particular the data transformation method is approximately 8 times faster than the so-called 2.5D reconstruction algorithm (see figure 2), which was introduced 1976 in [2] and is established in geoelectrics. Additionally, we succesfully applied this idea to a geophysical model and also in combination with the Factorization Method.

Conclusion & Future Study

1. The proposed idea represents a bridge between 3D and 2D forward problem.
2. Based on presented data transformation a novel approach is introduced for solving a cylindric 3D inverse problem approximately.
3. Potentially this data transformation can be applied to a wide class of inverse elliptic boundary value problems.
4. A better estimate for $\|u_2 - T(u_3)\|_f$ is desirable.

References

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