A Numerical Solution Method for an Infinitesimal Elasto-Plastic Cosserat Model

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A micropolar extension to infinitesimal elasticity

- We present a geometrically linear generalized continua of Cosserat micropolar type in the elasto-plastic case.
- We postulate independent infinitesimal microrotations of the material. Thus, as a consequence of balance of angular momentum, Cauchy stresses σ are not symmetric any more.
- Cosserat effects regularize the mesh size dependence of localization computations where shear failure mechanisms play a dominant role.
- We restrict Cosserat microrotations to the elastic response of the material. Inelasticity is formulated as in Prandtl-Reuß plasticity. The elasto-plastic Cosserat problem is well-posed (Neff/Chełmiński Appl.Math.Opti06, PRSE05).

Infinitesimal time continuous Cosserat Plasticity

Let $\Omega \subset \mathbb{R}^3$ be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary.

We consider infinitesimal microrotations $\overline{A} \in \mathfrak{so}(3) := \{\overline{B} \in \mathbf{R}^{3,3} : \overline{B}^T = -\overline{B}\}$, and we define the symmetric bilinear form

$$
\begin{aligned}
& a\left((\mathbf{u}, \overline{A}, \varepsilon_p), (\mathbf{v}, \overline{B}, \eta)\right) = \\
& 2\mu \int_\Omega \left(\operatorname{sym}(D\mathbf{u} - \varepsilon_p) : \operatorname{sym}(D\mathbf{v} - \eta)\right) d\mathbf{x} + \lambda \int_\Omega \operatorname{tr} D\mathbf{u} \cdot \operatorname{tr} D\mathbf{v} \, d\mathbf{x} \\
& + 2\mu_c \int_\Omega \left(\operatorname{skew}(D\mathbf{u} - \overline{A}) : \operatorname{skew}(D\mathbf{v} - \overline{B})\right) d\mathbf{x} + 2\mu L_c^2 \int_\Omega D\overline{A} : D\overline{B} \, d\mathbf{x} \,,
\end{aligned}
$$

where μ_c is the Cosserat couple modulus and L_c the internal length scale.

Here, no physical interpretation of Cosserat parameters!

The goal is regularization through elastic size effect.

Furthermore, we have the convex functional

$$j(\varepsilon_p) = \int_\Omega K_0 |\varepsilon_p| d\mathbf{x} \ ,$$

(with $\varepsilon_p \in \{\eta \in \mathbf{R}^{d,d} \colon \eta^T = \eta \ , \mathrm{tr}\, \eta = 0\}$), and the load functional

$$\ell(t, \mathbf{v}) = \int_\Omega \mathbf{b}(t) \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v} \, d\mathbf{a} \ .$$

Quasi-static infinitesimal Cosserat plasticity is characterized by the variational inequality

$$a\Big((\mathbf{u}, \overline{A}, \varepsilon_p), (\mathbf{v}, \overline{B}, \eta) - (\dot{\mathbf{u}}, \dot{\overline{A}}, \dot{\varepsilon}_p)\Big) + j(\eta) - j(\dot{\varepsilon}_p) \geq \ell(\mathbf{v} - \dot{\mathbf{u}}) \ ,$$
$$\text{for all } (\mathbf{v}, \overline{B}, \eta) \in \mathbf{V} \times W \times \mathbf{E} \ ,$$

subject to boundary conditions and given material history $\varepsilon_p(0) = \varepsilon_p^0$ as initial value, where $\mathbf{V} \times W \times \mathbf{E}$ is a suitable function space.

Infinitesimal Elasto-Plastic Cosserat Model - Equations

We want to determine

displacements	$\mathbf{u}$:	$\overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}^3$,
infinitesimal micro-rotations	$\overline{A}$:	$\Omega \times [0, T] \longrightarrow \mathfrak{so}(3)$,
non-symmetric stresses	$\boldsymbol{\sigma}$:	$\Omega \times [0, T] \longrightarrow \mathbf{R}^{3,3}$,
symmetric plastic strains	ε_p:	$\Omega \times [0, T] \longrightarrow \mathrm{Sym}(3)$ with $\varepsilon_p(0) = \mathbf{0}$,
and a plastic multiplier	Λ:	$\Omega \times [0, T] \longrightarrow \mathbb{R}$,

satisfying the essential boundary conditions and the equilibrium equations

$$\begin{aligned}
-\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}, t) &= \mathbf{b}(\mathbf{x}, t), && (\mathbf{x}, t) \in \Omega \times [0, T]\,, \\
\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x}, t), && (\mathbf{x}, t) \in \Gamma_N \times [0, T]\,, \\
-\mu L_c^2 \Delta \overline{A}(\mathbf{x}, t) &= \mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x}, t)) - \overline{A}(\mathbf{x}, t)\big), && (\mathbf{x}, t) \in \Omega \times [0, T]\,, \\
D\overline{A}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{0}, && (\mathbf{x}, t) \in \Gamma_N \times [0, T],
\end{aligned}$$

the constitutive relation

$$\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x}, t) &= 2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x}, t)) - \varepsilon_p(\mathbf{x}, t)\big) + \lambda \operatorname{tr} D\mathbf{u}(\mathbf{x}, t) \cdot \mathbf{I} \\
&\quad + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x}, t)) - \overline{A}(\mathbf{x}, t)\big), \qquad (\mathbf{x}, t) \in \Omega \times [0, T]\,,
\end{aligned}$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

the complementary conditions for the yield criterion

$$\Lambda(\mathbf{x},t)\phi\big(T_E(\mathbf{x},t)\big) = 0, \quad \Lambda(\mathbf{x},t) \geq 0, \quad \phi\big(T_E(\mathbf{x},t)\big) \leq 0, \quad (\mathbf{x},t) \in \Omega \times [0,T] \,.$$

and the flow rule

$$\frac{d}{dt}\varepsilon_p(\mathbf{x},t) = \Lambda(\mathbf{x},t) D\phi\big(T_E(\mathbf{x},t)\big) \,, \qquad (\mathbf{x},t) \in \Omega \times [0,T] \,,$$

depending on $T_E(\mathbf{x},t) = 2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x},t)) - \varepsilon_p(\mathbf{x},t)\big)$.

For given material history $\varepsilon_p(t)$ at fixed time t, the displacement and the micro-rotations are determined by minimizing the total elastic energy

$$\mathcal{I}(\mathbf{u},\overline{A},\varepsilon_p) = \mathcal{E}(\varepsilon(\mathbf{u}),\overline{A},\varepsilon_p) - \ell(t,\mathbf{u}) \,,$$

$$\begin{aligned} \text{with } \mathcal{E}(\varepsilon,\overline{A},\varepsilon_p) \quad &= \quad \mu\int_\Omega \left|\operatorname{sym}(\varepsilon) - \varepsilon_p\right|^2 d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \operatorname{tr}(\varepsilon)^2\, d\mathbf{x} \\ &\quad + \mu_c \int_\Omega |\operatorname{skew}(\varepsilon) - \overline{A}|^2\, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\overline{A}|^2\, d\mathbf{x} \,. \end{aligned}$$

Discrete formulation of the Elasto-Plastic Model

Let $\mathbf{V}_h \times W_h$ be a finite element space with $\mathbf{v}_h = \mathbf{0}$ and $\overline{B}_h = 0$ on Γ_D.
The model of incremental infinitesimal plasticity is obtained by a decomposition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval and backward Euler scheme.

Lemma:

The fully discrete elasto-plastic problem is equivalent to the following nonlinear weak problem. For given ε_p^{n-1} find $(\mathbf{u}^n, \overline{A}^n) \in \mathbf{V}_h \times W_h$ such that

$$\int_\Omega P_{\mathbf{K}}\big(2\mu(\mathrm{sym}(D\mathbf{u}_h^n) - \varepsilon_p^{n-1})\big) : D\mathbf{v}_h \, d\mathbf{x} + \lambda \int_\Omega \mathrm{tr}\, D\mathbf{u}_h^n \cdot \mathrm{tr}\, D\mathbf{v}_h \, d\mathbf{x}$$
$$+ 2\mu_c \int_\Omega \big(\mathrm{skew}(D\mathbf{u}_h^n) - \overline{A}_h^n\big) : D\mathbf{v}_h \, d\mathbf{x} = \ell(t_n, \mathbf{v}_h), \qquad \mathbf{v}_h \in \mathbf{V}_h \,,$$

$$\mu L_c^2 \int_\Omega D\overline{A}_h^n \cdot D\overline{B}_h \, d\mathbf{x} = \mu_c \int_\Omega \big(\mathrm{skew}(D\mathbf{u}_h^n) - \overline{A}_h^n\big) : \overline{B}_h \, d\mathbf{x}, \qquad \overline{B}_h \in W_h \,,$$

with the orthogonal projection $P_{\mathbf{K}}(\boldsymbol{\theta}) = \boldsymbol{\theta} - \max\{0, |\mathrm{dev}(\boldsymbol{\theta})| - K_0\} \frac{\mathrm{dev}(\boldsymbol{\theta})}{|\mathrm{dev}(\boldsymbol{\theta})|}$
on the elastic domain $\mathbf{K} := \{\boldsymbol{\tau} \in \mathbf{R}^{3,3} : \boldsymbol{\tau}^T = \boldsymbol{\tau} \,, |\mathrm{dev}\, \boldsymbol{\tau}| \le K_0\}$
of the von Mises flow rule.

Variational Formulation of the discrete problem

Lemma:
Any minimizer $(\mathbf{u}_h^n, \overline{A}_h^n) \in \mathbf{V}_h \times W_h$ of the functional

$$\mathcal{I}_{\mathrm{incr}}^n(\mathbf{u}_h, \overline{A}_h) = \mathcal{E}_{\mathrm{incr}}(D\mathbf{u}_h, \overline{A}_h, \varepsilon_p^{n-1}) - \ell(t_n, \mathbf{u}_h)$$

solves the nonlinear variational update problem. Here $\mathcal{E}_{\mathrm{incr}}$ denotes the free energy of the incremental update problem defined by

$$\begin{aligned}\mathcal{E}_{\mathrm{incr}}(D\mathbf{u}, \overline{A}, \varepsilon_p) \quad = \quad & \frac{1}{2\mu} \int_\Omega \psi_{\mathbf{K}}\big(2\mu(\mathrm{sym}(D\mathbf{u}) - \varepsilon_p)\big)\, d\mathbf{x} + \frac{\lambda}{2} \int_\Omega \mathrm{tr}(D\mathbf{u})^2\, d\mathbf{x} \\ & + \mu_c \int_\Omega |\,\mathrm{skew}(D\mathbf{u}) - \overline{A}|^2\, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\overline{A}|^2\, d\mathbf{x}\ ,\end{aligned}$$

$$\psi_{\mathbf{K}}(\boldsymbol{\theta}) \quad = \quad \begin{cases} \frac{1}{2}\,|\boldsymbol{\theta}|^2 & |\,\mathrm{dev}(\boldsymbol{\theta})| \le K_0\ , \\ \frac{1}{2}\left(\frac{1}{d}\ \mathrm{tr}(\boldsymbol{\theta})^2 + 2K_0\,|\,\mathrm{dev}(\boldsymbol{\theta})| - K_0^2\right) & |\,\mathrm{dev}(\boldsymbol{\theta})| > K_0\ . \end{cases}$$

If $\varepsilon_p^{n-1} = 0$ and $\mu_c = 0 \to$ classical Hencky-Problem.

The FEM convergence

Theorem:
We have

$$\|(\mathbf{u}-\mathbf{u}_h,\overline{A}-\overline{A}_h)\|_{\mathbf{V}\times W} \leq \frac{C}{\mu_c} \inf_{(\mathbf{v}_h,\overline{B}_h)\in\mathbf{V}_h\times W_h} \|(\mathbf{u}-\mathbf{v}_h,\overline{A}-\overline{B}_h)\|_{\mathbf{V}\times W} \,.$$

C is independent of $\mu_c \in (0,\mu]$.

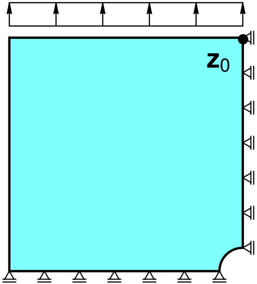
If the analytical solution u is H^2-smooth then

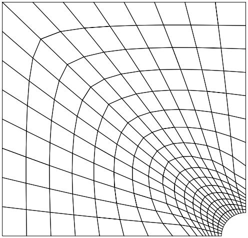
$$\inf_{(\mathbf{v}_h,\overline{B}_h)\in\mathbf{V}_h\times W_h} \|(\mathbf{u}-\mathbf{v}_h,\overline{A}-\overline{B}_h)\|_{\mathbf{V}\times W} \leq C\,h\,(\|u\|_{\mathbf{H}^2(\Omega)} + \|\overline{A}\|_{\mathbf{H}^2(\Omega)}) \,.$$

Neff/Knees: Cosserat update problem admits unique global H^2-solutions!
Idea: balance h against μ_c.

Plate with a hole

Let $\Omega = (0,10) \times (0,10) \setminus B_1(10,0)$. We use Q1 discretization and present results for 198147 unknowns on uniform refinement level 4. We have chosen the parameters $K_0 = 450, \lambda = 110743.8, \mu = 80193.8$ and $L_c = 0.020833$.
And apply traction force by Neumann boundary condition according to:
$\ell(t,v) = 100t \int_0^{10} \mathbf{v}(x_1, 10) dx_1$.

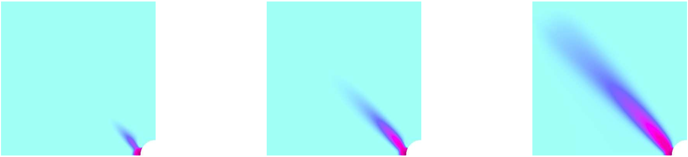




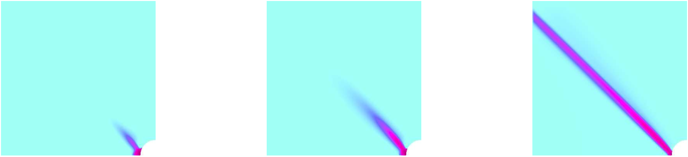
Geometry, boundary conditions and coarse mesh.

Numerical Experiment with M++

Cosserat Model ($\mu_c = \mu$) : Effective plastic strain

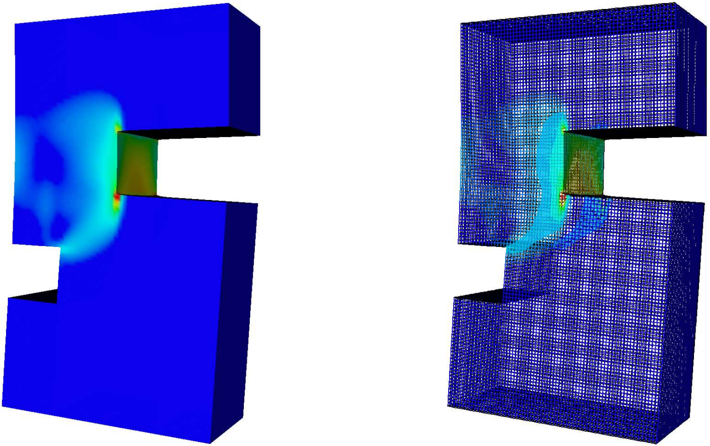


Prandtl-Reuß ($\mu_c = 0$) : Effective plastic strain



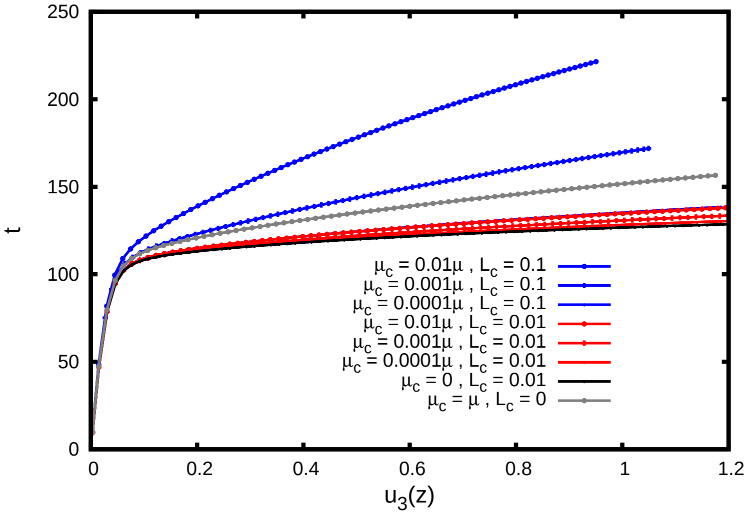
$t = 4.00$ $t = 4.40$ $t = 4.69$

Notched cube - a test configuration in 3D



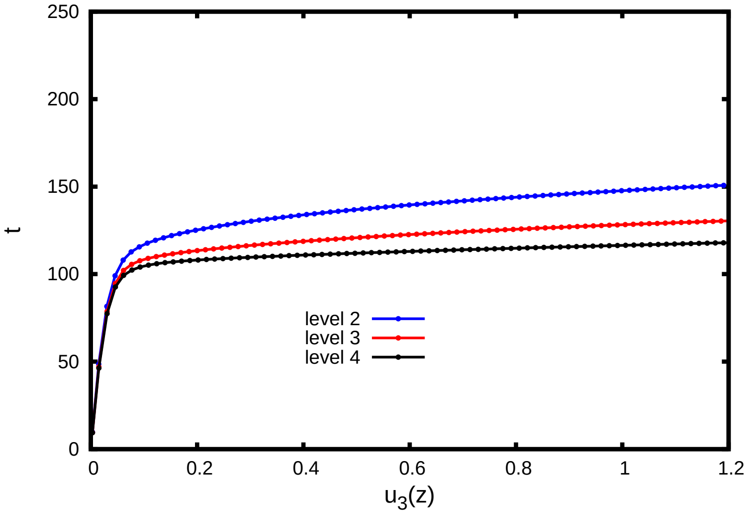
For the linear subproblem in each Newton step,
we use a parallel multilevel GMRES algorithm.

Numerical Experiment with M++



Load-displacement curve for notched cube on refinement level 3.
Here, we use 174.727 dof for displacement and microrotation.

Numerical Experiment with M++



Here, we compute up to 1.311.751 dof on refinement level 2, 3 and 4 ($\mu_c = 0.0001\mu$, $L_c = 0.01$). We observe about linear convergence in h.

Summary and Outlook

- The Elasto-Plastic Cosserat Model with pure Dirichlet data is well-posed: unique solution globally in time.
- The discrete update problem admits global H^2-solutions.
- For linear Lagrange elements, we have linear convergence in h.
- The Elasto-Plastic Cosserat Model is a regularization for classical perfect plasticity (shear failure mechanisms).
- Future work will be the analysis and robust implementation of geometrically nonlinear elasto-plastic Cosserat Models.

Selected References

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