A Numerical Solution Method for an Infinitesimal Elasto-Plastic Cosserat Model

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A micro-polar extension to infinitesimal elasticity

- We present a geometrically linear generalized continua of Cosserat micro-polar type in the elasto-plastic case.
- Starting with linear elasticity we postulate independent infinitesimal micro-rotations of the material. Thus, as a consequence of balance of angular momentum, stresses $\boldsymbol{\sigma}$ are not symmetric any more.
- Cosserat regularize the mesh size dependence of localization computations where shear failure mechanisms play a dominant role.
- Models are of engineering interest: Diebels/Ehlers, Iordache/William, Dietsche/Steinmann/Willam, Ristinmaa/Vecchi, de Borst, ...
- We restrict Cosserat micro-rotations to the elastic response of the material. Inelasticity is concerned as in Prandtl-Reuß plasticity. Thus, the elasto-plastic Cosserat problem is well-posed (Neff/Chełmiński).

Infinitesimal Elastic Cosserat Model - Data

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary. We fix a time interval $[0, T]$.

Given data:

displacement vector	$\mathbf{u}_D$:	$\Gamma_D \times [0, T] \longrightarrow \mathbb{R}^d$,
suitable infinitesimal micro-rotations	$\bar{A}_D$:	$\Gamma_D \times [0, T] \longrightarrow \mathfrak{so}(d)$,

where $\mathfrak{so}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} : \boldsymbol{\tau}^T = -\boldsymbol{\tau}\}$ is the Lie algebra of skew-symmetric matrices, and a load functional

$$\ell(t, \mathbf{v}) = \int_\Omega \mathbf{b}(t) \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v} \, d\mathbf{a}$$

depending on traction force densities $\mathbf{t}_N$ and body force densities $\mathbf{b}$.

The material is described by a linear elastic response depending on the Lamé constants $\lambda, \mu > 0$ and Cosserat constants $L_c > 0$ and $\mu_c \geq 0$.

Infinitesimal Elasto-Plastic Cosserat Model - Flow rule

Inelastic material behavior is modeled by a convex function

$$\phi\colon\ \mathrm{Sym}(d) \longrightarrow \mathbb{R}$$

determining the convex set of admissible stresses $\mathbf{K} = \{\boldsymbol{\tau} \in \mathrm{Sym}(d)\colon \phi(\boldsymbol{\tau}) \le 0\}$. We assume that ϕ is smooth for $\boldsymbol{\tau} \neq \mathbf{0}$, and we assume $\phi(\mathbf{0}) < 0$. In a first approach we choose the von Mises yield criterion $\phi(\boldsymbol{\tau}) = |\operatorname{dev}(\boldsymbol{\tau})| - K_0$ for a given constant $K_0 > 0$. Where we used $\operatorname{dev}\colon\ \mathrm{Sym}(d) \longrightarrow \mathrm{Sym}(d)$ with $\operatorname{dev}\boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}\operatorname{tr}\boldsymbol{\tau}$.

We have $P_{\mathbf{K}}(\boldsymbol{\theta}) = \boldsymbol{\theta} - \max\{0, \gamma\}\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}$.

We use the following realization $\mathbf{C}(\boldsymbol{\theta}) \in \partial P_{\mathbf{K}}(\boldsymbol{\theta})$, where $P_{\mathbf{K}}(\boldsymbol{\theta})$ is the multi-valued derivative of the projection defined by $\mathbf{C}(\boldsymbol{\theta}) = \mathrm{id}$ for $|\operatorname{dev}(\boldsymbol{\theta})| \le K_0$ and

$$\mathbf{C}(\boldsymbol{\theta}) = \frac{1}{d}\mathbf{I}\otimes\mathbf{I} + \frac{K_0}{|\operatorname{dev}(\boldsymbol{\theta})|}\Big(\big(\mathrm{id} - \frac{1}{d}\mathbf{I}\otimes\mathbf{I}\big) - \frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}\otimes\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}\Big) \text{ for } |\operatorname{dev}(\boldsymbol{\theta})| > K_0\,.$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

We want to determine

displacements	$\mathbf{u}:$	$\bar{\Omega} \times [0, T] \longrightarrow \mathbb{R}^d$,
infinitesimal micro-rotations	$\bar{A}:$	$\Omega \times [0, T] \longrightarrow \mathfrak{so}(d)$,
non-symmetric stresses	$\boldsymbol{\sigma}:$	$\Omega \times [0, T] \longrightarrow \mathbf{R}^{d,d}$,
symmetric plastic strains	$\varepsilon_p:$	$\Omega \times [0, T] \longrightarrow \mathrm{Sym}(d)$ with $\varepsilon_p(0) = \mathbf{0}$,
and a plastic multiplier	$\Lambda:$	$\Omega \times [0, T] \longrightarrow \mathbb{R}$,

satisfying the essential boundary conditions and the equilibrium equations

$$
\begin{aligned}
-\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}, t) &= \mathbf{b}(\mathbf{x}, t), && (\mathbf{x}, t) \in \Omega \times [0, T]\ , \\
\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x}, t), && (\mathbf{x}, t) \in \Gamma_N \times [0, T]\ , \\
-\mu L_c^2 \Delta \bar{A}(\mathbf{x}, t) &= \mu_c \big(\mathrm{skew}(D\mathbf{u}(\mathbf{x}, t)) - \bar{A}(\mathbf{x}, t) \big), && (\mathbf{x}, t) \in \Omega \times [0, T]\ , \\
D\bar{A}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{0}, && (\mathbf{x}, t) \in \Gamma_N \times [0, T],
\end{aligned}
$$

the constitutive relation

$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{x}, t) &= 2\mu \big(\mathrm{sym}(D\mathbf{u}(\mathbf{x}, t)) - \varepsilon_p(\mathbf{x}, t) \big) + \lambda \operatorname{div}(\mathbf{u})(\mathbf{x}, t)\mathbf{I} \\
&\quad + 2\mu_c \big(\mathrm{skew}(D\mathbf{u}(\mathbf{x}, t)) - \bar{A}(\mathbf{x}, t) \big), \qquad (\mathbf{x}, t) \in \Omega \times [0, T]\ ,
\end{aligned}
$$

Infinitesimal Elasto-Plastic Cosserat Model - Equations

the complementary conditions for the yield criterion

$$\Lambda(\mathbf{x},t)\phi\big(T_E(\mathbf{x},t)\big)=0,\quad \Lambda(\mathbf{x},t)\geq 0,\quad \phi\big(T_E(\mathbf{x},t)\big)\leq 0,\quad (\mathbf{x},t)\in\Omega\times[0,T]\ .$$

and the flow rule

$$\frac{d}{dt}\varepsilon_p(\mathbf{x},t)=\Lambda(\mathbf{x},t)D\phi\big(T_E(\mathbf{x},t)\big)\ ,\qquad (\mathbf{x},t)\in\Omega\times[0,T]\ ,$$

depending on $T_E(\mathbf{x},t)=2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x},t))-\varepsilon_p(\mathbf{x},t)\big)$.

For given material history $\varepsilon_p(t)$ at fixed time t, the displacement and the micro-rotations are determined by minimizing the total energy

$$\mathcal{I}(\mathbf{u},\bar{A},\varepsilon_p)=\mathcal{E}(\varepsilon(\mathbf{u}),\bar{A},\varepsilon_p)-\ell(t,\mathbf{u})\ ,$$

with
$$\begin{aligned}\mathcal{E}(\varepsilon,\bar{A},\varepsilon_p) &= \mu\int_\Omega\big|\operatorname{sym}(\varepsilon)-\varepsilon_p\big|^2\,d\mathbf{x}+\frac{\lambda}{2}\int_\Omega\operatorname{tr}(\varepsilon)^2\,d\mathbf{x}\\ &\quad+\mu_c\int_\Omega|\operatorname{skew}(\varepsilon)-\bar{A}|^2\,d\mathbf{x}+\mu L_c^2\int_\Omega|D\bar{A}|^2\,d\mathbf{x}\ .\end{aligned}$$

Discretization in space

Let h be a mesh size parameter, and let $\mathbf{V}_h \subset C^{0,1}(\Omega, \mathbb{R}^d)$ and $W_h \subset C^{0,1}(\Omega, \mathfrak{so}(d))$ be finite element spaces, and set

$$\mathbf{V}_h(\mathbf{u}_D) = \{\mathbf{v} \in \mathbf{V}_h : \mathbf{v}(\mathbf{x}) = \mathbf{u}_D(\mathbf{x}) \text{ for } \mathbf{x} \in D_h\},$$
$$W_h(A_D) = \{B \in W_h : B(\mathbf{x}) = A_D(\mathbf{x}) \text{ for } \mathbf{x} \in D_h'\}$$

where $D_h, D_h' \subset \Gamma_D$ are the sets of all nodal points on Γ_D for $\mathbf{u}$ and A.

Let $\boldsymbol{\Xi}_h \subset \Omega$ be quadrature points and let $\omega_{\boldsymbol{\xi}}$ be corresponding quadrature weights such that

$$\int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} = \sum_{\boldsymbol{\xi} \in \boldsymbol{\Xi}_h} \omega_{\boldsymbol{\xi}} \, \mathbf{v}(\boldsymbol{\xi}) \cdot \mathbf{w}(\boldsymbol{\xi}) \,, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{V}_h \;.$$

We set

$$\begin{aligned} \boldsymbol{\Lambda} &= \{\Lambda \colon \boldsymbol{\Xi}_h \longrightarrow \mathbb{R}\} \,, \\ \boldsymbol{\Sigma}_h &= \{\boldsymbol{\tau} \colon \boldsymbol{\Xi}_h \longrightarrow \mathbf{R}^{d,d}\} \,, \\ \text{and } \mathbf{E}_h^p &= \{\boldsymbol{\tau} \colon \boldsymbol{\Xi}_h \longrightarrow \mathfrak{sl}(d) \cap \mathrm{Sym}(d)\} \,, \end{aligned}$$

where $\mathfrak{sl}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} \colon \operatorname{tr}(\boldsymbol{\tau}) = 0\}$ is the Lie algebra of trace-free matrices.

Discretization in space

Determine

displacements $\mathbf{u}$:	$[0,T] \longrightarrow \mathbf{V}_h$,
stresses $\boldsymbol{\sigma}$:	$[0,T] \longrightarrow \boldsymbol{\Sigma}_h$,
micro-rotations $\bar{A}$:	$[0,T] \longrightarrow W_h$,
plastic strains ε_p:	$[0,T] \longrightarrow \mathbf{E}_h^p$,
and a plastic multiplier Λ:	$[0,T] \longrightarrow \boldsymbol{\Lambda}$

satisfying

- the equilibrium equations,

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma} : D\mathbf{v}\, d\mathbf{x} &= \ell(\cdot,\mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \;, \\ \mu L_c^2 \int_\Omega D\bar{A} \cdot D\bar{B}\, d\mathbf{x} &= \mu_c \int_\Omega \left(\mathrm{skew}(D\mathbf{u}) - \bar{A}\right) : \bar{B}\, d\mathbf{x}, \;\; \bar{B} \in W_h(\mathbf{0}) \end{aligned}$$

- the essential boundary conditions,
- the constitutive relation,
- the complementary conditions (Kuhn-Tucker),
- and the flow rule.

Discretization in time

The model of incremental infinitesimal plasticity is obtained by a decomposition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval and backward Euler scheme. For $n = 1, 2, 3, \ldots$ the next increment depends on the material history described by ε_p^{n-1}, the new load $\ell^n[\mathbf{v}] = \ell(t_n, \mathbf{v})$, and the new Dirichlet boundary values $\mathbf{u}_D^n = \mathbf{u}_D(t_n)$ and $\bar{A}_D^n = \bar{A}_D(t_n)$.

We compute the displacement vector $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, the stresses $\boldsymbol{\sigma}^n \in \boldsymbol{\Sigma}_h$, the micro-rotations $\bar{A} \in W_h(\bar{A}_D^n)$, the plastic strains $\varepsilon_p^n \in \mathbf{E}_h^p$, and the plastic multiplier $\Lambda^n \in \boldsymbol{\Lambda}$ satisfying additionally the discrete flow-rule:

$$\frac{1}{t_n - t_{n-1}} \Big(\varepsilon_p^n(\xi) - \varepsilon_p^{n-1}(\xi) \Big) = \Lambda^n(\xi) D\phi\big(T_E^n(\xi) \big) \ , \qquad \xi \in \boldsymbol{\Xi}_h \ ,$$

depending on $T_E^n(\xi) = 2\mu\big(\operatorname{sym}(D\mathbf{u}^n(\xi)) - \varepsilon_p^n(\xi) \big)$.

Since the problem is rate-independent, we define $\gamma^n = (t_n - t_{n-1})\Lambda^n \in \boldsymbol{\Lambda}$.

Fully discrete elasto-plastic problem

Together, we can state the fully discrete elasto-plastic Cosserat problem.

For given $\varepsilon_p^{n-1} \in \mathbf{E}_h^p$ find $\boldsymbol{\sigma}^n$, $T_E^n \in \boldsymbol{\Sigma}_h$, $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, $\bar{A} \in W_h(\bar{A}_D^n)$ and $\gamma^n \in \boldsymbol{\Lambda}$ such that

$$T_E^n(\xi) = 2\mu\Big(\operatorname{sym}(D\mathbf{u}^n(\xi)) - \varepsilon_p^{n-1}(\xi) - \gamma^n(\xi) D\phi\big(T_E^n(\xi)\big)\Big)\ , \qquad \xi \in \boldsymbol{\Xi}_h\ ,$$

$$\phi\big(T_E^n(\xi)\big) \le 0\ , \quad \gamma^n(\xi)\phi\big(T_E^n(\xi)\big) = 0\ , \quad \gamma^n(\xi) \ge 0\ , \qquad \xi \in \boldsymbol{\Xi}_h\ ,$$

$$\boldsymbol{\sigma}^n(\xi) = T_E^n(\xi) + \lambda \operatorname{div}(\mathbf{u}^n)(\xi)\mathbf{I} + 2\mu_c(\operatorname{skew}(D\mathbf{u}^n(\xi)) - \bar{A}^n(\xi))\ , \quad \xi \in \boldsymbol{\Xi}_h\ ,$$

$$\int_\Omega \boldsymbol{\sigma}^n : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}]\ , \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}),$$

$$\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} = \mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\, d\mathbf{x}\ , \qquad \bar{B} \in W_h(\mathbf{0}).$$

Discrete formulation of the Elasto-Plastic Cosserat Model

Lemma:
The fully discrete elasto-plastic problem is equivalent to the following nonlinear variational problem. For given ε_p^{n-1} find $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n)$ such that

$$\int_\Omega P_K\big(2\mu(\mathrm{sym}(D\mathbf{u}^n) - \varepsilon_p^{n-1})\big) : D\mathbf{v}\, d\mathbf{x} + \lambda \int_\Omega \mathrm{div}(\mathbf{u}^n)\,\mathrm{div}(\mathbf{v})\, d\mathbf{x}$$
$$+\, 2\mu_c \int_\Omega \big(\mathrm{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \,,$$

$$\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} = \mu_c \int_\Omega \big(\mathrm{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\, d\mathbf{x}, \qquad \bar{B} \in W_h(\mathbf{0}) \,.$$

Variational Formulation of the discrete problem

Lemma:
Any minimizer $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n)$ of the functional

$$\mathcal{I}_{\rm incr}^n(\mathbf{u}, \bar{A}) = \mathcal{E}_{\rm incr}(\varepsilon(\mathbf{u}), \bar{A}, \varepsilon_p^{n-1}) - \ell_n[\mathbf{u}]$$

solves the nonlinear variational problem. Here $\mathcal{E}_{\rm incr}$ denotes the free energy of the incremental problem defined by

$$\begin{aligned}\mathcal{E}_{\rm incr}(D\mathbf{u}, \bar{A}, \varepsilon_p) &= \frac{1}{2\mu}\int_\Omega \psi_{\mathsf{K}}\big(2\mu(\mathrm{sym}(D\mathbf{u}) - \varepsilon_p)\big)\, d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \mathrm{tr}(D\mathbf{u})^2\, d\mathbf{x} \\ &\quad + \mu_c \int_\Omega |\,\mathrm{skew}(D\mathbf{u}) - \bar{A}|^2\, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\bar{A}|^2\, d\mathbf{x}\ .\end{aligned}$$

and a convex, non-negative, potential

$$\psi_{\mathsf{K}}(\boldsymbol{\theta}) = \frac{1}{2}\ |\boldsymbol{\theta}|^2 - \frac{1}{2}\ |\boldsymbol{\theta} - P_{\mathsf{K}}(\boldsymbol{\theta})|^2\ .$$

for $\boldsymbol{\theta} \in \mathrm{Sym}(d)$.

Numerical Solution Algorithm

We formulate a semi-smooth Newton method for the nonlinear variational problem

$$(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n): \qquad F^n(\mathbf{u}^n, \bar{A}^n) = 0$$

in every time step n, where F^n is the first variation of $\mathcal{I}_{\mathrm{incr}}^n$ defined by

$$F^n(\mathbf{u}, \bar{A})[\mathbf{v}, \bar{B}] = D\mathcal{I}_{\mathrm{incr}}^n(\mathbf{u}, \bar{A})[\mathbf{v}, \bar{B}], \qquad (\mathbf{v}, \bar{B}) \in \mathbf{V}_h(\mathbf{0}) \times W_h(\mathbf{0}) \ .$$

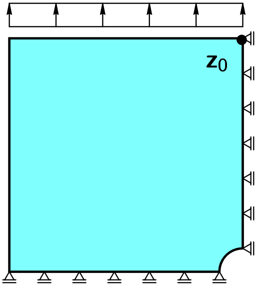
The functional F^n is semi-smooth, and the second variation $\partial^2\mathcal{I}_{\mathrm{incr}}^n = \partial F^n$ is multi-valued. Thus, the corresponding semi-smooth Newton method can be formally written as

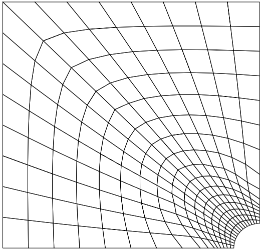
$$0 \in F^n(\mathbf{u}^{n,k}, \bar{A}^{n,k}) + \partial F^n(\mathbf{u}^{n,k}, \bar{A}^{n,k})[\mathbf{u}^{n,k+1} - \mathbf{u}^{n,k}, \bar{A}^{n,k+1} - \bar{A}^{n,k}] \ .$$

We consider the special case of the von Mises flow rule. The Newton increment is realized using the consistent linearization $\mathbf{C}(\theta)$.

Benchmark problem for parameter study of μ_c

Let $\Omega = (0, 10) \times (0, 10) \setminus B_1(10, 0)$. We use Q1 discretization and present results for 198147 unknowns on uniform refinement level 4. We have chosen the parameters $K_0 = 450, \lambda = 110743.8, \mu = 80193.8$ and $L_c = 0.020833$.
And apply traction force by Neumann boundary condition according to:
$\ell(t, v) = 100t \int_0^{10} \mathbf{v}(x_1, 10) dx_1$.

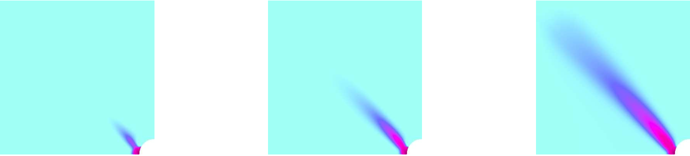




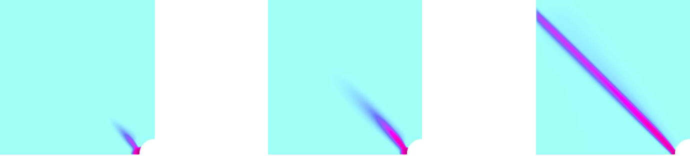
Geometry, boundary conditions and coarse mesh for the benchmark problem.

Numerical Experiment with M++

Cosserat Model ($\mu_c = \mu$) : Effective plastic strain

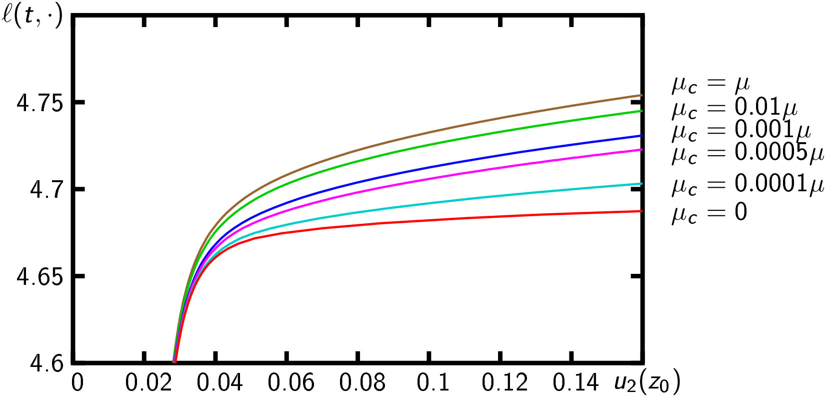


Prandtl-Reuß ($\mu_c = 0$) : Effective plastic strain



$t = 4.00$ $t = 4.40$ $t = 4.69$

Numerical Experiment with M++



Load-displacement curve: displacement $\mathbf{u}$ is evaluated at special point $\mathbf{z}_0$.

Summary and Outlook

- The Elasto-Plastic Cosserat Model with pure Dirichlet data is well-defined. Solution exists global in time.
- Elasto-Plastic Cosserat Model is a regularization for classical plasticity.
- Complete finite element analysis (dependent on μ_c) is available and will be published.
- Future work will be the analysis and implementation of nonlinear Cosserat Models.